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Stochastic Duels
Comprehensive Compendium of Results

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report is the second and final half of a project to exhaustively review the literature of one-on-one stochastic duels. This part contains a comprehensive compendium of all results contained in papers which were previously listed in the annotated bibliography of the preceding report. The results are given in a common notation and are systematically organized.

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A7. ANCKER, C.J., Jr., "The Stochastic Duel with Time-Dependent Hit Probabilities," University of Southern California, Los Angeles, California, ISE Department Technical Report TR 79-2, November, 1979, 18pp.

FM - CRIFT - time-dependent hit probabilities, $p(t)$

$\underline{\text{General Solutions}}$: Integral equations for, $h^O(t)$, $h(t)$, $\Theta_O(u)$, and $\Phi(u)$. If cf of $p(t)$ has one pole (not necessarily simple) in the lower-half of the complex plane, then $\Theta_O(u)$ and $\Phi(u)$ are given.

$\qquad \underline{\text{Example}}.$ $X = \text{ned}(r)$, $q(t) = \eta\, e^{-\rho t}$

$\qquad\qquad h^O(t)$, $h(t)$.

FD - CRIFT - Same as FM for each contestant.

$\underline{\text{General Solution}}$: $P(A)$

$\qquad \underline{\text{Example}}$ (1). $X_A = \text{ned}(r_A)$, $X_B = \text{ned}(r_B)$

$\qquad\qquad q_A(t) = \eta\, e^{-\rho t}$, $q_B(t) = \xi\, e^{-\zeta t}$

$\qquad\qquad P(A)$

$\qquad \underline{\text{Example}}$ (2). Same as (1), except q_B constant

$\qquad\qquad P(A)$

renewal theory
integral equations
characteristic functions

B-11a

FOREWORD

This work is a continuation of a project whose first two parts were reported in a document entitled, "The One-on-One Stochastic Duel: Parts I and II," a U. S. Army Research Office Interim Technical Report under Contract Number DAAG 29-78-C-0012, dated 15 April 1979.

The first report contained, among other things, a bibliography of all research materials on this subject, known to the author. This report summarizes the results contained in the papers in the prior bibliography. The results are given in a common notation and systematically organized (to aid in locating a desired result) in a comprehensive compendium.

This work completes the project. The prior issue and this volume should be viewed as a single work and consulted simultaneously.

Immediately preceding this foreward is a page which should be inserted in "Part II - An Annotated Bibliography," immediately following page B-11.

The author would be grateful to learn of any inadvertent mistakes, omissions, or other errors which may have occured.

LIST OF SYMBOLS, NOTATION AND DEFINITIONS

1. ## WORD AND PHRASE DEFINITIONS

Burst
: - A group of rounds fired consecutively in a cluster. The groups are separated in some systematic manner. Usually refers to automatic weapon fire where time between rounds in a group may be taken as a constant

Fundamental Marksman
: - Refers to the basic one-on-zero (or Marksman versus a passive target) problem. The marksman fires at his target until he gets a hit which terminates the process. The time between rounds (interfiring times) is either a specified constant or a specified random variable and is the same from round to round. On each round fired, he may hit his target with a specified probability which is the same from round to round. The interfiring times and hit probabilities are independent from round to round. He starts at time zero with unlimited ammunition and unlimited time to fire and fires his first round at the next firing time.

Fundamental Duel
: - Two fundamental marksman (see definition above) fire at each other as targets until the first hits, which

terminates the process in a win for the successful contestant.

Salvo — A firing mode in which a number of weapons are fired in sequence at a constant interval.

Volley — A firing mode in which a number of weapons are fired simultaneously.

2. ENGLISH ALPHABET

A
- one of the two contestants in the duel, usually used as a subscript
- event, a kill by A

a
- the factor a, when a_1/b_1 reduces to a/b (ratio of relatively prime integers)
- time between rounds in a burst
- arbitrary constant

a_1
- A's fixed interfiring time (a_1/b_1 is rational)

AB
- event, neither A nor B wins the duel (i.e., a draw occurs)

B
- one of the two contestants in the duel, usually used as a subscript
- event, a kill by B

b
- the factor b, when a_1/b_1 reduces to a/b (ratio of relatively prime integers)
- arbitrary constant

b_1
- B's fixed interfiring time (a_1/b_1 is rational)

C
- arbitrary constant
- function of constants

CRIFT
- Continuous Random Interfiring Times

CRV
- Continuous Random Variable

c
- superscript denoting complementary

c_i
- constants

cdf   - complementary distribution function $\triangleq \int_{t}^{\infty} f(x)dx$, denoted

by superscript c, i.e., F^c

cf   - characteristic function $\triangleq \int_{-\infty}^{\infty} e^{iut} f(t)dt$

crv   - continuous random variable

D   - rv, damage inflicted by a single round

d   - value of D

           - P[round in volley kills | volley hits] (fixed)

df   - distribution function $\triangleq \int_{-\infty}^{t} f(x)dx = F_X(t)$

E_j - the j-th event for A or M

           - the j-th state of A or M

E_M - the killed state

E_0 - the acquisition state (available as a target)

E[X]   - expectation of the rv, X

Erlang(k ; r) - a rv with pdf

$$f_X(x) = \frac{(kr)^k x^{k-1}}{(k-1)!} e^{-krx} ; \quad x > 0, \quad k = 1,2,\ldots$$

$$= 0 \quad\quad\quad ; \quad \text{elsewhere}$$

and with cf $\phi_X(u) = \dfrac{(rk)^k}{(rk - iu)^k}$

F_j - the j-th event for B

           - the j-th state of B

$F_X(t)$ - df of any rv, X

$F_X^c(t)$ – cdf of any rv, X

FD   – the Fundamental Duel problem (defined on page iii)

FIFT   – Fixed Interfiring Times

FM   – The Fundamental Marksman problem (defined on page iii)

$f_A(x)$ – pdf of rv, X_A

$f_B(x)$ – pdf of rv, X_B

$f_X(x)$ – pef of <u>any given</u> rv X , e.g., specifically, the rv's

           T_F, T_G, T_L, T_R, T_S, and X

$f^{(n)}(t)$ $- \dfrac{d^n(f(t))}{dt^n}$

$G_X(z)$ – one-variable geometric transform of rv, X. (z-transform)

$G_X(z, w)$ – two-variable geometric transform of rv, X. (zw-transform)

$g(t)$ – pdf of rv, T_D

$g_A(t)$ – pdf of rv, $T_{D \mid A}$

$g_{AB}(t)$ – pdf of rv, $T_{D \mid AB}$

$g_B(t)$ – pdf of rv, $T_{D \mid B}$

$g_0(t)$ – improper pdf of rv, T_O

gt   – geometric transform $\triangleq \Sigma_{all\ n}\ f(n)z^n$, also sometimes called

           the z-transform; and if $f(n)$ are elements of a pmf,

           sometimes called a probability generating function

H   – event, a hit

H_i – event, a hit on the i-th round fired

$\overline{H}$ – event, no hit

$\overline{H}_i$ — event, failure to hit on i-th round fired

$H(t)$ — df of rv, T_M

$H^c(t)$ — cdf of rv, T_M

$h(t)$ — pdf of rv, T_M

$h_A(t)$ — pdf of rv, T_A

$h_B(t)$ — pdf of rv, T_B

$h_{A1}(t)$ — pdf of rv, $T_{A,H}$ (same as $h_A(t)$ if no limitations)

$h_{B1}(t)$ — pdf of rv, $T_{B,H}$ (same as $h_B(t)$ if no limitations)

$h_H(t)$ — pdf of rv, $T_{M|H}$

$h_{\overline{H}}(t)$ — pdf of rv, $T_{M|\overline{H}}$

$h_K(t)$ — pdf of rv, T_K

$h_O(t)$ — pdf of rv, $T_{M,\overline{H}}$

$h_1(t)$ — pdf of rv, $T_{M,H}$ (same as $h(t)$ if no limitations)

$h^O(t)$ — improper pdf of rv, $T_{\overline{H}}$

$h^1(t)$ — improper pdf of rv, $T_{\overline{H}}^i$

I — rv, initial number of rounds A or M has

— also, same as above, fixed

I^+ — set of positive integers

$I_x(m,n)$ — the Incomplete Beta Function Ratio

$$\triangleq \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)} \int_0^x \xi^{m-1}(1-\xi)^{n-1}\,d\xi$$

$$= 1 - (1-x)^n \sum_{k=0}^{m-1} \binom{n+k-1}{k} x^k$$

IFT   - Interfiring Time

i   - imaginary number $(\sqrt{-1})$

      - an arbitrary constant

      - a summation index

iid   - independent, identically distributed (used in reference to two or more rv's)

J   - rv, initial number of rounds B has

      - also, same as above, fixed

j   - value of J

      - a summation index

K   - event, a kill

      - rv, round number on which A or M has a failure

$\bar{K}$ - event, no kill

k   - values of rv, K

      - an arbitrary constant

      - a summation index

k_A - P[A kills | a near miss by A]

k_j - arbitrary constants

L   - rv, round number on which B has a failure

LT   - Laplace Transform $= \int_0^\infty e^{-st} f(t)dt$

ℓ - value of rv, L

      - arbitrary constant

M	- symbol denoting the marksman
	- rv, initial supply of weapons
	- a constant $\overset{\triangle}{=}$ a + b
MIFT	- Mixed Interfiring Times, i.e., one side with IFT a rv, the other side IFT constant
MLE	- Maximum Likelihood Estimate
m	- value of rv, M
	- fixed number of initial weapons
	- a constant $\overset{\triangle}{=}$ [a/b]
	- number of states in a Markov firing process
	- an arbitrary constant
mgf	- moment generating function, defined as the same as the Laplace Transform with s (in LT) replaced by - s
N	- rv, total number of rounds fired
	- an arbitrary constant
$N(\mu, \sigma^2)$	- a normally distributed rv with mean μ, and variance σ^2
n	- value of rv, N
	- an arbitrary constant
	- a summation index
n_0	- an arbitrary, fixed, round number
n_1	- a constant $\overset{\triangle}{=}$ [ℓb/a]
ned	- negative exponential rv with pdf

$$f_X(x) = re^{-rx}, \quad x, r > 0$$
$$= 0, \quad \text{elsewhere}$$

x

with cf $\phi_X(u) = r/(r - iu)$

$P(E)$ or $P[E]$ - probability of the event, E

$P(A)_f$ - $P(A)$ for the fundamental duel

$P(A)_U$ - $P(A)$ for the unlimited (fundamental) duel with ned IFT's

for X_A and X_B

p - hit probability, constant on each round fired

- constant $\stackrel{\Delta}{=} P[H_i]$

p_{An} - P[hit by A on n-th round | n-th round fired]

p_a - probability of acquiring the target

p_{ij} - P[going from state E_i to state E_j]

p_j - P[of being in state E_j initially]

P_k - P[K on next round | H on last round]

p_{n1} - P[hit on n-th round | n-th round is fired]

P_O - $P[H_i | H_{i-1}]$

P_1 - first round hit probability

- $P[H_i | \bar{H}_{i-1}]$

pdf - probability density function, e.g., $f_X(x)$

pgf - probability generating function (sometimes called geometric

transform, or z-transform)

pmf - probability mass function, e.g., $p_X(x)$

p(t) - hit probability (as a function of time since start)

p(x) - hit probability (as a function of time since last firing)

$p_X(x)$ - pmf of the rv, X

q   - miss probability $= 1 - p$ (also, for all the subscripted p's
             above)

$q(t)$ - miss probability as a function of time since start $= 1 - p(t)$

$q(x)$ - miss probability as a function of time since last firing $=$
             $1 - p(x)$

R   - rv, number of hits to a kill (used where more than one hit is
             required to kill)

             - same as above, except a fixed value

r   - value of rv, R

             - rate of fire $\stackrel{\triangle}{=}$ E[X]

rv   - random variable

$r(t)$ - rate of fire (as a function of time) for non-stationary
             Poisson process

$\mathcal{R}$ - remainder when a is divided by $b = a - b[a/b]$, $0 \leq \mathcal{R} < b$

s   - probability of a near miss

T   - rv, marksman's time to a firing (any number of previous
             firings)

             - rv, marksman's time between bursts (only when firing in
             bursts)

T_A - rv, A's time to a kill (unopposed)

$T_{A,H}$ - rv, A's time to a kill (unopposed) - used in situations where
             A can run out of time or ammunition, etc.

T_B - rv, B's time to a kill (unopposed)

T_b - rv, time since beginning of either duel or marksman's firing

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$T_{B,h}$ - rv, (similar to $T_{A,H}$)

T_C - rv,   time during which marksman is in contact and firing

          - rv,   time since start of the duel and opponents are in contact and firing

$T_{\bar{C}}$ - rv,   time during which marksman is not in contact (not firing), or may not be a target

          - rv,   time since start of the duel and opponents are not in contact (not firing)

T_D - rv,   time-duration of the duel

$T_{D\,|\,A}$ - rv,   time-duration of the duel, given A wins

$T_{D\,|\,AB}$ - rv,   time-duration of the duel given a draw occurs

$T_{D\,|\,B}$ - rv,   time-duration of the duel given B wins

T_{d_A} - rv,   time for A to displace and resume firing

T_E - rv,   time during which ammunition is exhausted

T_F - rv,   time-of-flight

T_G - rv,   time between rounds in a burst

T_H - rv,   the event $[t < T < t + dt\,,\,H]$

$T_{\bar{H}}$ - rv,   the event $[t < T < t + dt\,,\,\bar{H}]$

$T_{\bar{H}}^i$ - rv,   the event $[t < T < t + dt\,,\,i$ hits, no kill$]$

T_K - rv,   rv,   time to fire the killing round

T_L - rv,   time-limitation on the firing

          - rv,   time to weapon failure

T_M - rv,   time to end of marksman's firing

$T_{M\,|\,H}$ - rv,   time to end of marksman's firing, given a hit

$T_{M,H}$ - rv,   time to end of marksman's firing, and a hit

$T_{M|\bar{H}}$ - rv,   time to end of marksman's firing, given no hit

$T_{M,\bar{H}}$ - rv,   time to end of marksman's firing, and no hit

T_0 - rv,   time to an event in the _duel_ with _no_ kill

T_R - rv,   time to fire R hits

T_S - rv,   sighting time, i.e., time during which A fires and for some specified reason B does not (e.g., A is concealed); (negative values mean that B has the advantage)

T_W - rv,   time until next supply replenishment of ammunition

t - time, values of the various rv's above

t_J - fixed time to go from state E_J to _any_ other state

t_s - value of rv T_S

        - fixed sighting time

$U(x)$ - unit step function $\overset{\triangle}{=} 1, \quad x \geq 0$

$$= 0, \quad x < 0$$

u - transforms variable in cf

        - $P[H_i | \bar{H}_{i-1}]$

$V[X]$ - variance of rv, X

v - probability of a weapon failure

        - fixed number of rounds fired in a volley by A

        - velocity

w - transform variable in cf

        - fixed number of rounds fired in a volley by B

        - variable of integration

X	$-$ rv, interfiring time for M
X_A	$-$ rv, interfiring time for A
X_B	$-$ rv, interfiring time for B
X_C	$-$ rv, time in contact
$X_{\bar{C}}$	$-$ rv, time not in contact
X_s	$-$ rv, searching time
x	$-$ value of the rv X
	$-$ an arbitrary constant
$[x_j]$	$-$ a constant $\triangleq [(j + 1)(\rho/b)]$
Y	$-$ discrete rv, number of rounds A fires before B acquires A (initial surprise)
	$-$ rv, time since last event (backwards recurrence time)
y	$-$ value of rv, Y
	$-$ an arbitrary constant
y_C	$-$ time since the event, the last contact
$y_{\bar{C}}$	$-$ time since the event, last lost contact
y_s	$-$ time since the event, last searching period started
z	$-$ constant, number of rounds in a burst
	$-$ transform variable for gt

2. GREEK ALPHABET

α	$-$ arbitrary constant
β	$-$ arbitrary constant

$\Gamma(y)$ — The Gamma Function

$$\triangleq \int_0^\infty \xi^y e^{-\xi} d\xi = (y-1)! \quad (\text{if } y \in I^+)$$

γ — arbitrary constant

$\gamma(x,y)$ — The Incomplete Gamma Function

$$\triangleq \int_0^x \xi^y e^{-\xi} d\xi$$

Δ — increment, e.g., Δt = increment of variable t

$\delta(x-a)$ — Dirac Delta Function

ε — arbitrary constant

ζ — arbitrary constant

η — arbitrary constant

$\Theta(u)$ — cf of $f_{T_L}(t)$

$\Theta_{\bar{C}}(u)$ — cf of $f_{T_{\bar{C}}}(t)$

$\Theta_1(u)$ — cf of $h^1(t)$

$\Theta_R(u)$ — cf of $f_{T_R}(t)$

$\Theta_S(u)$ — cf of $f_{T_S}(t)$

$\Theta_0(u)$ — cf of $h^0(t)$

$\varkappa_n$ — n-th cumulant of $h_H(t)$

κ_n — n-th cumulant of $f_X(x)$

λ — cycle time in fixed IFT duels = $a_1 b = a b_1$

— arbitrary constant

— characteristic value

$\lambda(y)$ — this is $\triangleq \dfrac{f_X(y)}{F_X^c(y)}$, and also

$$f_X(y) = \lambda(y) \, e^{-\int_0^y \lambda(\xi)d\xi}$$

μ — $E[X]$

$\mu_n(A)$ — n-th moment of $g_A(t)$ about the origin

$\mu_n(AB)$ — n-th moment of $g_{AB}(t)$ about the origin

$\mu_n(H)$ — n-th moment of $h_H(t)$ about the origin

$\mu_n(\bar{H})$ — n-th moment of $h_{\bar{H}}(t)$ about the origin

ν — summation index

ξ — variable of integration

    — summation index

ρ — fixed time interval between bursts

    — corr $[H_i, H_{i-1}]$

    — arbitrary constant

ρ_A — $P[A$ scores a near miss]

ρ_{An} — $P[$hit by A on the n-th round, n-th round fired]

ρ_n — $P[$hit by marksman on n-th round, n-th round fired]

σ^2 — $V[X]$

τ — arbitrary time constant, e.g., fixed time limit for duration
of duel or marksman's firing

    — variable of integration

$\phi(u)$ — cf of $h(t)$

$\Phi_A(u)$ — cf of $h_A(t)$

$\Phi_{A1}(u)$ — cf of $h_{A1}(t)$

$\Phi_K(u)$ — cf of $h_K(t)$

$\Phi_O(u)$ — cf of $h_O(t)$

$\Phi_1(t)$ — cf of $h_1(t)$

$\phi(u)$ — cf of $f_X(x)$

$\phi_F(u)$ — cf of $f_{T_F}(t)$

$\phi_G(u)$ — cf of $f_{T_G}(t)$

$\psi_A(u)$ — cf of $g_A(t)$

$\psi_{AB}(u)$ — cf of $g_{AB}(t)$

$\psi_H(u)$ — cf of $h_H(t)$

$\psi_O(u)$ — cf of $g_O(t)$

$\psi(u,y)$ — jointly, the pdf of the time since the last event, y,

and the cf of the DF of the time since the commencement

of marksman's firing (or the duel)

$\Omega(u)$ — cf of $q(t)$

  — cf of $q(x)$

ω — variable of integration

4. <u>MATRIX AND VECTOR NOTATION</u>

$\underset{\sim}{A}$ — $m \times n$ matrix with components a_{ij}

$\underset{\sim}{A}^T$ — $\underset{\sim}{A}$ transpose

$\underset{\sim}{A}^{-1}$ — $\underset{\sim}{A}$ inverse

$(\underset{\sim}{A})^{\nu}$ — $(a_{11}\, a_{12}\, \cdots\, a_{1n},\, a_{21}\, a_{22}\, \cdots\, a_{2n},\, \ldots,\, a_{m1}\, a_{m2}\, \cdots\, a_{mn})^{T}$, a column vector of mn components from the $\underset{\sim}{A}$ matrix. N.b., if $\underset{\sim}{A}$ is known $(\underset{\sim}{A})^{\nu}$ may be written down and vice versa

$\underset{\sim}{A} \times \underset{\sim}{B}$ — Kronecker product, i.e., a matrix whose ij-th component is $a_{ij}\underset{\sim}{B}$

$\underset{\sim}{a}$ — m component column vector

$\underset{\sim}{a}^{T}$ — $\underset{\sim}{a}$ transpose, a row vector

$\underset{\sim}{D}(a_{i})$ — an $m \times n$ diagonal matrix,

$\underset{\sim}{e}_{n}^{T}$ — $(1,1,\ldots,1)$, n component, row vector of all ones

$e^{\underset{\sim}{A}}$ — the exponential matrix,

$$\triangleq \left(\frac{\underset{\sim}{I}}{0!} + \frac{\underset{\sim}{A}}{1!} + \frac{\underset{\sim}{A}^{2}}{2!} + \cdots + \frac{\underset{\sim}{A}^{i}}{i!} + \cdots \right)$$

$\underset{\sim}{I}$ — identity matrix of appropriate size

$\underset{\sim}{i}_{0}^{T}$ — initial state probability vector $\triangleq (p_{0}, p_{1}, \ldots, p_{m})$

$\underset{\sim}{m}^{T}$ — a vector of appropriate size $\triangleq (1,0,\ldots,0)$

$\underset{\sim}{n}^{T}$ — a vector of appropriate size $\triangleq (0,0,\ldots,1)$

$\underset{\sim}{P}$ - stochastic submatrix of $\underset{\sim}{S}$ for transitions from transient to transient states

$\underset{\sim}{p}^T$ - a vector $\overset{\triangle}{=} (1-p, p \vdots 0)$

$\underset{\sim}{q}^T$ - a stochastic subvector of $\underset{\sim}{p}^T \overset{\triangle}{=} (1-p, p)$

$\underset{\sim}{r}^T$ - interfiring time vector $\overset{\triangle}{=} (t_0, t_1, \ldots, t_{m-1})$

$\underset{\sim}{S}$ - state transition matrix

$\underset{\sim}{t}$ - stochastic subvector of $\underset{\sim}{S}$ for transitions from transient states to kill state

$\underset{\sim}{\lambda}^T$ - characteristic value vector $\overset{\triangle}{=} (\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots, \lambda_m)$

5. OTHER MATHEMATICAL NOTATION

$\approx$ - is approximately equal to

$\overset{\triangle}{=}$ - is defined to be equal to

$*$ - convolution operation $\overset{\triangle}{=} \int_0^t f(t-\xi) f(\xi)d\xi = f(t) * f(t)$

$j*$ - number of iterated convolutions of a function with itself, e.g., $f(t) * f(t) * f(t) = f^{3*}(t)$

$\sim$ - is distributed as

$[x]$ - largest integer less than or equal to x

$\langle x \rangle$ - max (largest integer less than x, 0)

$\binom{m}{n}$ - binomial coefficient $\overset{\triangle}{=} \dfrac{m!}{n!(m-n)!}$

$\hat{}$ - when placed over a symbol, denotes its maximum likelihood estimate, e.g., $\hat{p} =$ MLE of p

$|$ - symbol for conditionality, e.g., $A\,|\,B$ means the event A, given the event B

$\Rightarrow$ - symbol meaning implies that, e.g., $A \Rightarrow B$ means that A implies B

$O(\Delta t)$ - order of $\Delta t \Rightarrow \lim\limits_{\Delta t \to 0} \dfrac{O(\Delta t)}{\Delta t} = 0$

$f^{(n)}(x)$ - $\dfrac{\partial^n (f(x))}{\partial x^n}$

$\big|_{x=0}$ - means evaluated at $x = 0$, e.g., $f(x)\big|_{x=0} = f(0)$

$(P)\int$ - the Cauchy principal value of an improper integral

$\int_L$ - same as $\int_{-\infty - i\varepsilon}^{\infty - i\varepsilon}$ where ε is finite but less than the distance to the nearest singularity in the lower-half of the complex plane; this integration may be around any of the contours in the lower-half of the complex plane shown, as appropriate or convenient, and as $R \to \infty$
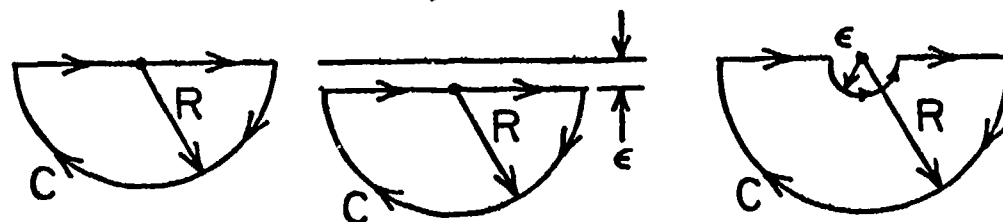
the integral on C must $\to 0$ as $R \to \infty$; the first path is only useful if there is no singularity on the real line

$\int_U$ - same as $\int_{-\infty + i\varepsilon}^{\infty + i\varepsilon}$ where ε is finite but less than the distance to the nearest singularity in the upper-half of the

complex plane; the integration may be around any of the contours
in the upper-half of the complex plane shown, as appropriate or
convenient, and as R → ∞
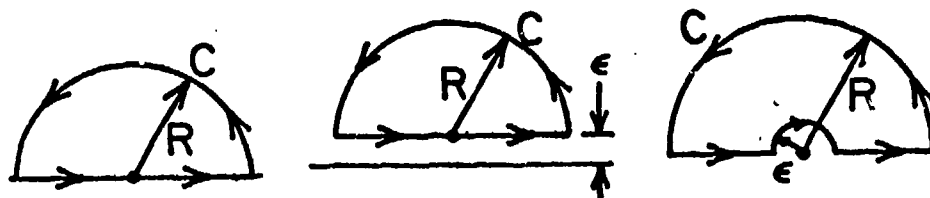
the integral on C must → 0 as R → ∞; the first path
is only useful if there is no singularity on the real line

PART III - A COMPREHENSIVE COMPENDIUM

OF RESULTS

TABLE OF CONTENTS   Page

INTRODUCTION

In this compendium, the notations in the page margins, such as A & W1, refer to the sources of the result. These sources are all listed in the preceding annotated bibliography. For example, A & W1 means the first paper in the bibliography by Ancker and Williams.

An asterisk (*) in the margin means that the result is new and given here for the first time.

A double asterisk (**) in the margin means a result for which a major error in the original manuscript has been corrected.

All marginal notations apply to the result given on the line on which the notation appears and to all preceding results up to the next marginal notation.

$P(B)$ will usually be omitted as it is easily derived from $P(B) = 1 - P(A)$ if a draw is impossible or from $P(B) = 1 - P(A) - P(AB)$ if a draw is possible.

THE FUNDAMENTAL MARKSMAN PROBLEM - FIXED INTERFIRING TIMES

(FM - FIFT)

I.   <u>FM-FIFT</u>

$X = a_1$

$P(H) = 1$

$P(T_M = na_1) = pq^{n-1}, \quad n = 1,2,\ldots$

$P[N = n] = pq^{n-1}, \quad n = 1,2,\ldots$

$E[N] = \dfrac{1}{p}, \quad E[N^2] = \dfrac{1+q}{p^2}$

$\left.\phantom{\begin{array}{c}1\\1\\1\end{array}}\right\}$ independent of a_1

$P[N \geq n_0] = q^{n_0-1}$

A & G1

II.   <u>LIMITED AMMUNITION</u>

     A.   <u>I a RV</u>

$X = a_1, \quad P[I = i] = \alpha_i, \quad P[I = \infty] = \alpha_\infty, \quad \alpha_\infty + \displaystyle\sum_{i=0}^{\infty} \alpha_i = 1$

$P(H) = p \displaystyle\sum_{n=1}^{\infty} q^{n-1} \left(\displaystyle\sum_{i=n}^{\infty} \alpha_i + \alpha_\infty \right)$ *

$P(\overline{H}) = \displaystyle\sum_{i=1}^{\infty} q^i \alpha_i$ *

$P(H)\, P(T_M = na_1 \mid H) = pq^{n-1}\left[\displaystyle\sum_{i=n}^{\infty} \alpha_i + \alpha_\infty \right]$ *

$P(\overline{H})\, P(T_M = na_1 \mid \overline{H}) = q^n \alpha_n$ *

$$P(N = n \mid H) = \frac{1}{P(H)} \, pq^{n-1}\left(\alpha_\infty + \sum_{i=n}^{\infty} \alpha_i\right), \qquad n \geq 1 \ ,$$

$$E[N \mid H] = \frac{1}{P(H)} \left[\frac{1 - \sum\limits_{i=0}^{\infty} \alpha_i q^i}{p} - \sum_{i=1}^{\infty} i \, \alpha_i q^i\right] \ ,$$

$$E[N^2 \mid H] = \frac{1}{P(H)} \left[\frac{1+q}{p}\left(1 - \sum_{i=0}^{\infty} \alpha_i q^i\right) - \frac{2}{p}\sum_{i=1}^{\infty} i \, \alpha_i q^i \right.$$
$$\left. - \sum_{i=1}^{\infty} i^2 \, \alpha_i q^i \right] \ ,$$

$$P(N \geq n_0 \mid H) = \frac{1}{P(H)}\left[\alpha_\infty + \sum_{i=0}^{\infty} \alpha_{n_0+i}(1 - q^{i+1})\right] q^{n_0-1} \ ,$$

$$P(N = n \mid \bar{H}) = \frac{1}{P(\bar{H})} \, \alpha_n q^n \ , \quad n \geq 0 \ ,$$

$$P(N = n) = \alpha_0 \qquad\qquad , \quad n = 0$$

A & G1

$$= pq^{n-1}\left(\alpha_\infty + \sum_{i=n}^{\infty} \alpha_i\right) + \alpha_n q^n \ , \quad n \geq 1$$

B. **FIXED SUPPLY**

$$X = a_1, \quad \alpha_k = 1, \quad \alpha_\infty = \alpha_i = 0, \qquad i \neq k$$

* $$P(H) = 1 - q^k$$

$$P(\overline{H}) = q^k \qquad \qquad *$$

$$P(H)\ P(T_M = na_1 \mid H) = pq^{n-1}, \qquad n = 1,2,\ldots,k \qquad *$$

$$P(\overline{H})\ P(T_M = na_1 \mid \overline{H}) = k\,a_1\,q^k \qquad \qquad *$$

C. I EITHER a RV or a CONSTANT

$$P(T_M = na_1) = P(H)\ P(T_M = na_1 \mid H) + P(\overline{H})\ P(T_M = na_1 \mid \overline{H}) \qquad *$$

III. LIMITED TIME DURATION

A. T_L a RV WITH pdf f_{T_L}

$X = a_1$

$$P(H) = p\ \sum_{n=1}^{\infty} q^{n-1} \int_{na_1}^{\infty} f_{T_L}(t)\,dt$$

$$P(\overline{H}) = \sum_{n=0}^{\infty} q^n \int_{na_1}^{(n+1)a_1} f_{T_L}(t)\,dt$$

$$P(H)\ P(T_M = na_1 \mid H) = pq^{n-1} \int_{na_1}^{\infty} f_{T_L}(t)\,dt$$

$$P(\overline{H})\ h_{\overline{H}}(t) = q^{[t/a_1]}\ f_{T_L} \quad \text{where } [x] = \text{largest integer } \le x\ .$$

B. T_L A CONSTANT

$T_L = \tau;\ X = a_1\,[\tau/a_1]$

$$P(H) = 1 - q^{[\tau/a_1]}$$

$$P(\overline{H}) = q^{[\tau/a_1]}$$

C5

$$P(H) \, P(T_M = na_1 \mid H) \; = \; pq^{n-1}, \qquad n = 1,2,\ldots,[\tau/a_1]$$

$$P(\overline{H}) \, h_{\overline{H}}(t) \; = \; q^{[\tau/a_1]} \; \delta(t - \tau)$$

C.   **FOR T_L EITHER A RV OR A CONSTANT**

$$h(t) \; = \; P(H) \, P(T_H = na_1 \mid H) \, \delta(t - na_1) + P(\overline{H}) \, h_{\overline{H}}(t)$$

Example: Let $f_{T_L}(t) = \frac{1}{\tau} e^{-t/\tau}$, $X = a_1$, then

$$P(H) \; = \; \frac{pe^{-a_1/\tau}}{1 - qe^{-a_1/\tau}} \quad ,$$

$$P(\overline{H}) \; = \; \frac{1 - e^{-a_1/\tau}}{1 - qe^{-a_1/\tau}} \quad ,$$

$$P(H) \, P(T_M = na_1 \mid H) \; = \; \frac{p}{q} \, (qe^{-a_1/\tau})^n \quad ,$$

A & G2
$$P(\overline{H}) \, h_{\overline{H}}(t) \; = \; \frac{q^{[t/a_1]}}{\tau} \, e^{-t/\tau} \quad .$$

IV.   **TIME-OF-FLIGHT INCLUDED**

$X = a_1$

$T_F =$ rv time-of-flight

No delay between rounds fired

$T_K = rv$ time-to-fire killing round

$T_M =$ time-to-hit target

$T_F = \tau$,   a constant

$$p_{T_K}(na_1) = p_N(n) = P[T_K = na_1] = P(N = n)$$
$$= pq^{n-1}, \qquad n = 1,2,\ldots$$

$$p_{T_M}(na_1 + \tau) = P[T_M = na_1 + \tau] = pq^{n-1}, \quad n = 1,2,\ldots \qquad \text{A3}$$

V. MARKOV-DEPENDENT FIRE

E_j, $j = 0,1,2,\ldots,m$ states of the system

E_0 - acquisition state (starting state, i.e., available as a target

E_m - killed state (absorbing)

E_j - other arbitrary specified states , $j \neq 0,m$

$p_j = P[\text{being in state } E_j \text{ initially}]$

$p_{ij} = P[\text{going from state } E_i \text{ to } E_j]$,   transition probability

$\underset{\sim}{1}_0^T = (p_0, p_1, \ldots, p_{m-1} \mathbin{\vdots} , p_m) = (\underset{\sim}{m}^T \mathbin{\vdots} p_m)$ - initial state vector

In what follows:

$\underset{\sim}{m}^T = (1, 0, 0, \ldots, 0)$ - m components

$$
\underset{\sim}{S} = \left(
\begin{array}{cccc|c}
p_{00} & p_{01} & \cdots & p_{0,m-1} & p_{0,m} \\
p_{10} & p_{11} & \cdots & p_{1,m-1} & p_{1,m} \\
\cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot \\
p_{m-1,0} & p_{m-1,1} & \cdots & p_{m-1,m-1} & p_{m-1,m} \\
\hline
0 & 0 & \cdots & 0 & 1
\end{array}
\right)
$$

$$
= \left(
\begin{array}{c|c}
\underset{\sim}{p} & \underset{\sim}{t} \\
\hline
0 & 1
\end{array}
\right) , \quad \text{transition matrix}
$$

t_j - fixed time to go from state E_j to any other state,
$\quad$ $j = 0,1,2,\ldots,m-1$

$\underset{\sim}{r}^T = (t_0, t_1, \ldots, t_{m-1})$, interfiring time vector (times may be
$\quad$ different for each state)

Bal $\qquad E[T_M] = \underset{\sim}{m}^T (\underset{\sim}{I} - \underset{\sim}{P})^{-1} \underset{\sim}{r}$

A. $\underline{m = 3}$

$\quad$ Let

$\qquad R = rv$ number of hits to a kill

$\qquad p_k = P[K \mid H], \qquad q_k = 1 - p_k$

$$p_R(r) = p_k \, q_k^{r-1}, \qquad r = 1, 2, \ldots$$

$$= 0 \qquad , \qquad \text{elsewhere}$$

$N = rv \quad$ number of rounds to a kill

$p_1 = $ 1-st round hit probability, $\quad q_1 = 1 - p_1$

$$\left. \begin{array}{l} p = P[H_i \mid H_{i-1}], \quad q = 1 - p \\[2mm] u = P[H_i \mid \overline{H}_{i-1}] \end{array} \right\} \quad i \geq 2$$

$$\underset{\sim}{S} = \begin{array}{r} \\ \text{acquisition} \\[3mm] (\text{miss}) \ \overline{H} \\[3mm] (\text{hit, not killed}) \ H\overline{K} \\[3mm] (\text{kill}) \ K \end{array} \begin{array}{cccc} \text{acq} & \overline{H} & H\overline{K} & K \\ \left(\begin{array}{ccc|c} 0 & 1 - p_1 & p_1(1 - p_k) & p_1 p_k \\[3mm] 0 & 1 - u & u(1 - p_k) & u\, p_k \\[3mm] 0 & 1 - p & p(1 - p_k) & p\, p_k \\ \hline 0 & 0 & 0 & 1 \end{array} \right) \end{array}$$

$$\underset{\sim}{r} = \begin{pmatrix} t_a + t_1 + t_f \\[3mm] t_m + t_f \\[3mm] t_h + t_f \end{pmatrix}, \quad \text{see definitions below}$$

$$p_{N \mid R}(n \mid r) = p_1 \, p^{r-1}, \qquad n = r$$

$$= \left[p_1 \sum_{k=2}^{r} \binom{r-1}{k-1} p^{r-k} q^{k-1} u^{k-1} \binom{n-r-1}{k-2} (1-u)^{n-r-k+1} \right] \text{for}$$
$$\left. + \, q_1 \sum_{k=1}^{r} \binom{r-1}{k-1} p^{r-k} q^{k-1} u^{k} \binom{n-r-1}{k-1} (1-u)^{n-r-k} \right]$$

FM - FIFT

TABLE I

THE MARKOV DEPENDENT FIRING DISTRIBUTION: $F\{N \mid R = a + y\}$

p	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9
u	0.1	0.1	0.5	0.5	0.5	0.5	0.5	0.5	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9
p_1	0.9	0.9	0.1	0.1	0.5	0.5	0.9	0.9	0.1	0.1	0.1	0.5	0.5	0.9	0.9	0.9
r	5	5	5	10	5	10	5	10	5	10	15	5	10	5	10	15
y/a	4	34	4	9	4	9	4	9	4	9	14	4	9	4	9	14
1	590	003	066	039	328	194	590	349	066	039	023	328	194	590	349	206
2	033	003	310	194	237	194	164	194	558	349	217	426	349	295	349	309
3	030	002	222	188	161	167	100	145	272	331	302	180	262	089	192	247
4	028	002	150	160	105	132	060	104	081	179	235	051	128	021	077	140
5	026	002	097	125	067	099	036	072	019	071	131	011	048		025	063
6	024	002	061	093	041	071	021	049		023	059		015			024
7	022	002	038	066	025	049	012	032			022					008
8	020	002	023	046	015	033	007	021			007					
9	019	001	014	031	009	022		013								
10	017	001	008	021	005	015		008								
11	016	001	005	013		009		005								
12	014	001		009		006										
13	013	001		006												
14	012	001														
15	011															
16	010															
17	009															
18	009															
19	008															
20	007															
21	007															
22	006															
23	006															
24	005															
25	005															
26	004															
27	004															
28	004															
29	003															
30	003															

R & S1

TABLE I - (Continued)

THE MARKOV DEPENDENT FIRING DISTRIBUTION: $P\{N \mid R = a + y\}$

p	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.9	0.9
u	0.5	0.5	0.5	0.5	0.9	0.9	0.9	0.9	0.9	0.5	0.5	0.5	0.5	0.5	0.9	0.9	0.9	0.9	0.1	0.1
p_1	0.1	0.1	0.9	0.9	0.1	0.1	0.1	0.5	0.5	0.1	0.1	0.5	0.5	0.9	0.1	0.1	0.9	0.9	0.1	0.1
r	5	10	5	10	5	10	15	5	15	5	10	5	10	10	5	10	5	10	5	5
y/a	7	18	6	17	7	16	25	6	24	5	12	4	11	11	5	11	4	10	4	34
1	011	008	012	010	054	031	017	021	009	041	016	031	013	020	073	019	056	014	066	008
2	036	017	045	021	231	104	053	128	032	036	030	078	027	038	220	062	208	054	062	007
3	074	029	097	035	438	225	121	349	083	121	047	117	044	057	308	131	314	123	058	007
4	109	044	135	052	196	289	199	326	158	137	064	137	051	075	237	194	252	192	055	006
5	129	059	147	069	057	200	229	129	216	134	079	137	076	089	111	211	120	215	052	006
6	132	073	138	082	013	095	182	035	210	119	088	123	087	096	035	174	038	181	049	005
7	121	083	117	091		035	109		149	098	093	103	093	097	008	112	009	118	046	005
8	102	089	092	094		011	052		082	077	092	081	093	093		057		061	043	005
9	081	089	069	091			021		037	057	086	060	088	084		024		026	040	004
10	062	086	049	085			007		015	041	078	044	080	074		009		009	038	004
11	045	079	034	075						029	068	031	070	062					035	004
12	032	070	023	064						019	057	021	059	051					033	003
13	022	060	015	054						013	047	014	049	040					031	003
14	015	049	010	043						008	037	009	039	031					029	003
15	010	040	006	034						005	029	006	030	024					027	003
16	006	032		026							022		023	018					025	002
17	004	024		020							017		017	013					023	002
18		018		014							012		013	009					022	002
19		014		010							009		009	006					020	002
20		010		007							006		007	005					019	002
21		007		005							004		005						017	001
22		005		004									003						016	001
23		004																	015	001
24																			014	001
25																			013	001
26																			012	001
27																			011	
28																			010	
29																			009	
30																			009	

R & S1

$$n \geq r + j, \qquad j = 1,2,\ldots$$

$$G_{N \mid R}(z) = z^r \left[\frac{p_1 + q_1 u z}{1 - (1-u)z} \right] \left[\frac{p + q u z}{(1 - (1-u)z)^{r-1}} \right]^{r-1}$$

$$E[N \mid R] = \mu_{N \mid R} = r + \frac{q_1}{u} + \frac{q(r-1)}{u}$$

$$T_M \mid R = rv \quad \text{time to a kill, given } R = r$$

$$T_M \mid R = c_1 + c_2 \, r + c_3 [N \mid R]$$

where

$$c_1 = t_a + t_1 - t_h \qquad\qquad c_3 = t_m + t_f$$

$$c_2 = t_h - t_m \qquad\qquad c_4 = t_a + t_1 - t_m$$

and

t_a = acquisition time

t_1 = time-to-fire first round

t_h = time-to-fire, given last event was a hit

t_m = time-to-fire, given last event was a miss

t_f = time-of-flight

all constants

Bol

$$E[T_M \mid R] = c_1 + c_2 \, r + c_3 \, E[N \mid R]$$

$$= c_1 + c_3 \, \frac{(p - p_1)}{u} + \left[c_2 + c_3 + c_3 \left(\frac{q}{u}\right) \right] r$$

$$E[T_M] = \sum_{r=1}^{\infty} E[T_M \mid R] \, p_R(r)$$

$$= c_1 + c_3 \, \frac{(p - p_1)}{u} + \left[c_2 + c_3 + c_3 \left(\frac{q}{u}\right) \right] \frac{1}{p_k}$$

B. $\underline{m = 3}$ (SAME PROBLEM AS A) DIFFERENT APPROACH

$$G_N(z) = \frac{p_k \, p_1 \, z + \{p_k \, q_1 - p_k \, q_1(1-u) - p_k \, p_1(1-u)\}z^2}{1 - \{(1-u) + p\,q_k\}z + \{p(1-u)q_k + u\,q\}z^2}$$

$$= \frac{k_1 \, z + k_2 \, z^2}{k_3 + k_4 \, z + k_5 \, z^2}$$

$$
\begin{aligned}
p_N(n) &= k_1 &&, n = 1\\
&= k_5 - k_1 k_4 &&, n = 2\\
&= \beta_1 \, \lambda_1^n + \beta_2 \, \lambda_2^n &&, \lambda_1 , \lambda_2 \quad \text{real}\\
\text{or} \quad &= (\beta_1 + \beta_2 \, n)\lambda^n &&, \lambda_1 = \lambda_2 = \lambda \quad \text{real, where}
\end{aligned}
$$

1) λ_1, λ_2 are the roots of $\lambda^2 + k_4 \, \lambda + k_5 = 0$

2) β_1, β_2 are solutions of $\begin{cases} k_1 = \beta_1 \, \lambda_1 + \beta_2 \, \lambda_2 \\ k_5 - k_1 k_4 = \beta_1 \, \lambda_1^2 + \beta_2 \, \lambda_2^2 \end{cases}$, $\quad n \geq 2$

or $\quad = \gamma \, r^n \cos (n\Theta + \beta)$

when 3) λ_1, λ_2 are complex conjugate,

i.e., $\lambda_{1,2} = r(\cos \Theta \pm i \sin \Theta)$, (determined from 1) above)

and 4) γ and β are determined from

$$k_1 = \gamma \, r \cos (\Theta + \beta)$$

$$k_5 - k_1 k_4 = \gamma \, r^2 \cos (2\Theta + \beta) \ .$$

$$E[N] = \frac{u + p_k \, q_1 + q_k \, q}{u \, p_k}$$

$$V[N] = \frac{1}{p_k} + \left(\frac{p_k \, q_1 + q_k \, q}{p_k} \right) \left(\frac{2}{u^2} + \frac{1}{u} \right)$$

$$+ \frac{2 q_k}{p_k \, u^2} \left(u + q_1 \right) \left(u + q \right)$$

$$+ \frac{2 q_k^2}{p_k^2 \, u^2} \left(u + q \right)^2 - \frac{(u + p_k \, u_1 + q_k \, q)^2}{p_k^2 \, u^2}$$

Ki1 $\quad G_{T_M}(z) = \dfrac{p_k \, z^{c_4}[(1 - z^{c_3}) p_1 \, z^{c_3} + u \, z^{2c_3}}{1 - (1 - u)z^{c_3} + q_k \, z^{c_2}[p \, z^{c_3}(1 - z^{c_3}) + u \, z^{2c_3}]}$

C. MLE ESTIMATION OF p_1, u, AND p

Let $S_1, S_2, \ldots, S_k$ be k independent sequences of trials, each trial terminating on the r-th success. Define

n_{SS}^i = number of transitions $S \longrightarrow S$ (success-to-success) ,

in sequence S_i; similarly, n_{SF}^i and n_{FS}^i. Also

$$N_S = \begin{array}{l} \text{Number of the sequences which have} \\ \text{a success on the 1-st trial} \end{array} \quad .$$

$$\hat{p}_1 = \frac{N_S}{k} \quad ,$$

$$\hat{p} = \frac{\sum\limits_{i=1}^{k} n_{SS}^i}{\sum\limits_{i=1}^{k} (n_{SS}^i + n_{SF}^i)} \quad ,$$

$$\hat{u} = \frac{\sum\limits_{i=1}^{k} n_{FS}^i}{\sum\limits_{i=1}^{k} (n_{FS}^i + n_{FF}^i)} \quad ,$$

also,

R & S1
$$\mu_{N \mid R} = r + \frac{(1 - \hat{p}_1)}{\hat{u}} + \frac{(1 - \hat{p})(r - 1)}{\hat{u}} \quad .$$

D. FIFT

$$\underset{\sim}{r}^T = (a_1, a_1, a_1, \ldots, a_1) = \underset{\sim}{e}_m^T a_1 \quad ,$$

$$P[N = n] = P[T_M = na_1] = \underset{\sim}{m}^T \underset{\sim}{p}^{n-1} \underset{\sim}{t} \quad ,$$

$$E[N] = \underset{\sim}{m}^T (\underset{\sim}{I} - \underset{\sim}{P})^{-1} \underset{\sim}{e}_m \quad ,$$

Ba2
$$V[N] = \underset{\sim}{m}^T (\underset{\sim}{I} + \underset{\sim}{P})(\underset{\sim}{I} - \underset{\sim}{P})^{-2} \underset{\sim}{e}_m - [\underset{\sim}{m}^T (\underset{\sim}{I} - \underset{\sim}{P})^{-1} \underset{\sim}{e}_m]^2 \quad .$$

E. BURST FIRING

z   rounds per burst

a   time units between rounds in a burst

ρ time between bursts
$\left. \right\}$ all constants

Ba3
$$P[N = n] = \underset{\sim}{m}^T \underset{\sim}{p}^{n-1} \underset{\sim}{t} \quad .$$

VI.   MISCELLANEOUS RESULTS

FIXED TIME LIMIT - KILL PROBABILITY A NONDECREASING FUNCTION
OF IFT AND ROUND NUMBER

x's may be chosen by firer but are $\geq \beta$, a given constant

$$q_i = q_1 \prod_{j=2}^{i} [\alpha + (1 - \alpha)e^{-x_j}], \quad \alpha, q_1 \text{ given constants.}$$

$T_L = \tau \geq \beta$, a constant (measured from time 1-st round is fired).

The selection of x_1, x_2, etc., to obtain maximum probability of a kill in time τ follows. Maximum number of rounds which may be fired is $M = 1 + [\tau/\beta]$ where $[x]$ = largest integer contained in x.

$$P_H(n) = 1 - \prod_{i=1}^{n} q_i$$

λ = an unknown constant to be determined.

THEOREM: Optimal x_i's are

$$x_2 \geq x_3 \geq \cdots \geq x_n \geq \beta, \quad 2 \leq n \leq M$$

such that x_i's are determined as follows:

1. select a fixed n ,

2. solve

$$(n-1) \ln \left(\frac{1 - \alpha}{\alpha} \right) + \sum_{i=2}^{n} \ln \left(\frac{n-i+1}{\lambda} \right) = \tau$$

for λ ,

a. if $\lambda \leq 0$, set $x_n = \beta$ and return to 1, with n

replaced by $n-1$ and τ by $\tau - \beta$, and

b. if $\lambda > 0$, go to 3.

3. Compute,

$$x_i = \ln\left(\frac{1-\alpha}{\alpha}\right) + \ln\left(\frac{n-i+1}{\lambda} - 1\right), i = 1,2,\ldots,n ,$$

4. repeat process 1. through 3. for $n = 1,2,\ldots,M$,

5. for every n, compute $P_H(n)$, and

Frl   6. select n for max $P_H(n)$.

FM-CRIFT

THE FUNDAMENTAL MARKSMAN PROBLEM - CONTINUOUS RANDOM
INTER-FIRING TIMES

(FM - CRIFT)

I. <u>FM - CRIFT</u>

$P(H) = 1$

$$h(t) = p \sum_{j=0}^{\infty} q^j r^{(j+1)*}(t) = \frac{p}{2\pi} \int_{-\infty}^{\infty} \frac{\phi(u) e^{-itu}}{1 - q\phi(u)} \, du$$

$$G_N(\phi(u)) = \Phi(u) = \frac{p\phi(u)}{1 - q\phi(u)} \qquad \text{W \& A1}$$

$$\mu_n(H) = \frac{\Phi^{(n)}(0)}{i^n} \qquad \text{A \& G2}$$

$$H(t) = \frac{p}{2\pi i} \int_{-\infty}^{\infty} \frac{\phi(u)(1 - e^{-itu})}{[1 - q\phi(u)]u} \, du \qquad \text{W \& A1}$$

$$\Theta_0(u) = \frac{q\phi(u)}{1 - q\phi(u)} \qquad *$$

$$P(N = n) = pq^{n-1}, \qquad n = 1,2,\ldots$$

$$E[N] = \frac{1}{p} \quad \text{and} \quad E[N^2] = \frac{1+q}{p^2} \qquad \Big\} \quad \text{Independent of } X$$

$$P[N \geq n_0] = q^{n_0-1}. \qquad \text{A \& G1}$$

<u>Cumulants of Time-to-Hit (X) In Terms of Cumulants of IFT (κ)</u>

$$X_1 = \frac{1}{p} \kappa_1 = \frac{1}{p} \mu_1 = \frac{1}{p} \mu$$

$$X_2 = \frac{1}{p^2} (p\kappa_2 + q\kappa_1^2) = \frac{1}{p^2} (p\sigma^2 + q\mu^2)$$

$$X_3 = \frac{1}{p^3} [p^2\kappa_3 + 3pq\,\kappa_1\kappa_2 + q(1 + q)\kappa_1^3] =$$

$$= \frac{1}{p^3} [p^2 \kappa_3 + 3pq\mu\sigma^2 + q(1 + q)\mu^3]$$

W1
$$\kappa_4 = \frac{1}{p^4} [p^3 \kappa_4 + p^2 q(4\mu\kappa_3 + 3(\sigma^2)^2) + 6pq(1 + q)\mu^2\sigma^2$$

$$+ q(p^2 + 6pq + 6q^2)\mu^4]$$

Example 1:

$$X \sim n \, e \, d(r)$$

$$\Phi(u) = \frac{pr}{pr - iu}$$

$$h(t) = pre^{-prt}$$

W & A1
A & G1
$$H(t) = \int_0^t h(\xi)d\xi = 1 - e^{-prt} \ .$$
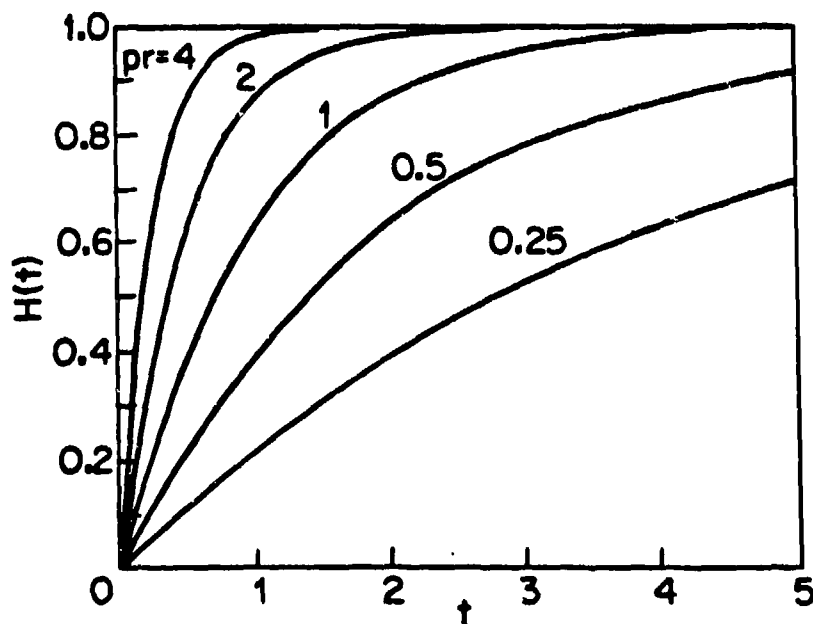
A6

$$\kappa_n = \frac{(n-1)!}{(pr)^n} \quad \text{and} \quad \kappa_n = \frac{(n-1)!}{r^n} . \qquad \text{W1}$$

Example 2 :

$X \sim \text{Erlang } (2,r)$

$$\Phi(u) = \frac{4 pr^2}{[(2r - iu)^2 - 4r^2 q]} \qquad \text{A\&G1}$$

$$h(t) = \frac{2pre^{-2rt}}{\sqrt{q}} \left[\frac{e^{2r\sqrt{q}\,t} - e^{-2r\sqrt{q}\,t}}{2} \right] = \frac{2pre^{-2rt}}{\sqrt{q}} \sinh 2r\sqrt{q}\,t$$

$$H(t) = 1 - \frac{e^{-2rt}}{\sqrt{q}} \left[\sinh 2r\sqrt{q}\,t + \sqrt{q} \cosh 2r\sqrt{q}\,t \right]$$
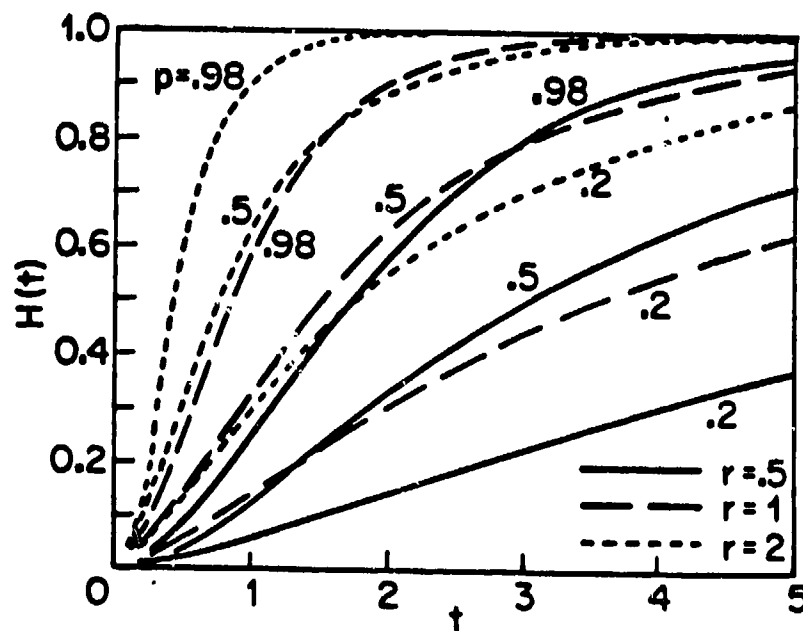
Example 3:

$X \sim$ Erlang (n, r)

$$\Phi(u) = \frac{(prn)^n}{(rn - iu)^n - (rn)^n q}$$

$$h(t) = \frac{npre^{-nrt}}{q^{(n-1)/n}}$$

$$\cdot \sum_{j=0}^{n-1} \frac{e^{nrtq^{1/n}e^{(i2\pi j)/n}}}{\prod_{\substack{k=0, k \neq j}}^{n-1} (e^{(i2\pi j)/n} - e^{(i2\pi k)/n})} \quad , \quad n = 2, 3, \ldots$$

A6

$$h(t) = \frac{npre^{-nrt}}{(2q^{1/n})^{n-1}}$$

$$\cdot \sum_{j=0}^{n-1} \frac{(-1)^j e^{(i2\pi j)/n} e^{nrtq^{1/n}e^{(i2\pi j)/n}}}{\prod_{\substack{k=0, k \neq j}}^{n-1} \sin \pi \frac{k-j}{n}} \quad , \quad n = 2, 3, \ldots$$

II. MULTIPLE HITS TO A KILL

A. R HITS TO A KILL - R FIXED

$$\Theta_0(u) = \frac{\phi(u) - \left(\frac{p\phi(u)}{1 - q\phi(u)} \right)^R}{1 - \phi(u)}$$

C23

$$\Psi(u,y) = [U(y) + \Theta_0(u)]e^{iuy - \int_0^y \lambda(\xi)d\xi}$$

$$\Phi(u) = \left[\frac{p\phi(u)}{1 - q\phi(u)} \right]^R$$

Example: Let $X \sim ned(r)$

$$h(t) \sim Erlang\ (R\,;pr)\ .$$

B. <u>R HITS TO A KILL - R a RV</u>

$$P[R = i] = \varepsilon_i, \qquad i = 1,2,\ldots$$

$$\Phi(u) = \sum_{i=1}^{\infty} \varepsilon_i \left[\frac{p\phi(u)}{1 - q\phi(u)} \right]^i$$

Example: Let $X \sim ned(r)$

$$\varepsilon_i = (1 - \varepsilon)\varepsilon^{i-1}$$

$$\Phi(u) = \frac{(1 - \varepsilon)pr}{pr(1 - \varepsilon) - iu} \implies h(t) \sim ned[(1 - \varepsilon)pr]\ . \qquad Bh\ 7$$

C. <u>LIMITED AMMUNITION - R HITS TO A KILL (R FIXED)</u>

$$P[X = i] = \alpha_i$$

$$\Phi_1(u) = \left[\frac{p\phi(u)}{1 - q\phi(u)} \right]^R \sum_{i=R}^{\infty} \alpha_i \left(1 - I_{q\phi(u)}(i - R + 1, R) \right)$$

$$P[\overline{K}] = \sum_{i=0}^{R-1} \alpha_i + \sum_{i=R}^{\infty} \alpha_i \sum_{j=0}^{R-1} \binom{i}{j} p^j q^{i-j}$$

$$h(t) = h_1(t) + P[\overline{K}]$$

$$h_1(t) = \sum_{i=R}^{\infty} \alpha_i \sum_{j=0}^{i-R} \binom{j + R - 1}{j} p^R q^j f^{(R+j)*}(t)$$

Example: Let $X \sim \text{ned}(r)$

$$\alpha_i = (1 - \alpha)\alpha^i, \qquad i = 0, 1, \ldots$$

$$P[\overline{K}] = 1 - \left(\frac{\alpha p}{1 - \alpha q} \right)^R$$

$$\Phi_1(u) = \left(\frac{\alpha p r}{r(1 - \alpha q) - iu} \right)^R$$

D. LIMITED AMMUNITION - R HITS TO A KILL - R a RV

$$P[I = i] = \alpha_i, \qquad i = 0, 1, 2, \ldots$$

Kw
& Kl

$$P[R = i] = \epsilon_i, \qquad i = 0, 1, 2, \ldots$$

$$\phi_1(u) = \sum_{j=1}^{\infty} \varepsilon_j \left[\frac{p\phi(u)}{1 - q\phi(u)} \right]^j \sum_{i=j}^{\infty} \alpha_i (1 - I_{q\phi(u)}(i - j + 1, j)) \; *$$

$$P[\bar{K}] = \sum_{j=1}^{\infty} \varepsilon_j \left[\sum_{i=0}^{j-1} \alpha_i + \sum_{i=j}^{\infty} \alpha_i \sum_{\nu=0}^{j-1} \binom{i}{\nu} p^{\nu} q^{i-\nu} \right] \qquad *$$

Example:

$$\alpha_i = (1 - \alpha)\alpha^i, \qquad i = 0,1,2,\ldots$$

$$\varepsilon_j = (1 - \varepsilon)\varepsilon^j, \qquad j = 0,1,2,\ldots$$

$$X \sim \text{ned}(r)$$

$$P[\bar{K}] = \frac{1 - \alpha}{1 - \alpha(1 - p\varepsilon)}$$

$$\phi_1(u) = \frac{\alpha \varepsilon \, p \, r}{r[1 - (1 - \varepsilon p)] - iu} \qquad\qquad \text{Kw \& Bl}$$

E. DAMAGE

$D = rv$, damage inflicted on a single round

1. Damage As a Function of Round Number

Let

rv total damage (in k rounds)

$$= D(k) \triangleq \sum_{i=1}^{k} D_i \quad \text{where} \quad D_i \text{ are iid and } \sim D$$

Let rv no. of rounds to total damage $< d$ $\equiv$ N(d).
Therefore

$$P[D(k) < d] \;=\; F_{D(k)}(d) \;=\; P_{N(d)}(k), \quad k = 0,1,2,\ldots$$

with $d \leq b$ (killing damage).

$$G_{N(d)}(z) \;=\; \sum_{k=0}^{\infty} z^k \, P_{N(d)}(k) \;.$$

Two Cases

a. **D is Discrete**

$$P[D = \ell] = p_D(\ell), \quad \sum_{\ell=0}^{\infty} p_D(\ell) = 1, \quad \ell \,\epsilon\, I^+,$$

$$G_{N(d)}(z) = 1 + z \sum_{\ell=0}^{d-1} p_D(\ell) \, G_{N(d-1)}(z) \;.$$

b. **D is Continuous**

$$P[y < D < y + dy] \;=\; f_D(y)dy$$

$$G_{N(d)}(z) = 1 + z \int_0^d G_{N(d-y)}(z) \, f_D(y)dy$$

For Either Case

$$\phi(u) \;=\; 1 - [1 - \phi(u)] \, G_{N(b)}[\phi(u)] \;.$$

Example: $X \sim \text{ned}(r)$

$$p_D(0) = \alpha , \quad p_D(1) = \beta , \quad p_D(b) = \gamma , \quad \alpha + \beta + \gamma = 1$$

$$\Phi(u) = 1 + \frac{iu}{(iu - r\gamma)} \left[\left\{ \frac{r\beta}{r(1 - \alpha) - iu} \right\}^b - 1 \right]$$

2. Damage is Time-Homogeneous

P[Marksman fires and increases damage to target by amount

   ℓ in time $[t, t + \Delta t]$ = $p_D(\ell)\Delta t + O(\Delta t)$

where

$$\sum_{\ell=1}^{\infty} p_D(\ell) = p_D , \quad 0 < p_{r_j} < 1 , \quad \ell \, \epsilon \, I^+ .$$

(n.b. probability of firing and probability of damage are
both included here. Non-stationary Poisson process).

$D(t) = rv$, total damage - up to time t (integer-valued)

$F_{D(t)} \overset{\Delta}{=} P[D(t) < d]$, $\quad d \leq b$ (killing damage)

Let

$$G_D(z) = \sum_{\ell=1}^{\infty} z^{\ell-1} p_D(\ell) ,$$

$$G_{D(t)}(z) = \sum_{d=1}^{b} z^{d-1} F_{D(t)}(d) = \frac{1}{1 - z} e^{t[zG_D(z) - p_D]}$$

$$h(t) = \underset{z^{b-1}}{coeff} \text{ in the } z \text{ expansion of } \left[\frac{p_D - zG_D(z)}{1 - z} \right]$$

$$\cdot \; e^{-[p_D - zG_D(z)]t}$$

or

$$\phi(u) = \underset{z^{b-1}}{coeff} \text{ in the } z \text{ expansion of } \left[\frac{p_D - zG_D(z)}{1 - z} \right]$$

$$\cdot \; [p_D - zG_D(z) - iu]^{-1} \; .$$

3. **Damage is a Function of Round Number and is Time-Dependent**

P[Marksman fires in $(t, t + \Delta t) \,|\, n$ rounds fired previously]

$$= r_n(t)\Delta t + 0(\Delta t)$$

P[Marksman fires n rounds in $(0, t)$ = $P_N(n; t)$

n.b. essentially, a non-stationary Poisson process

for each round fired:

P[Miss] = α ,   P[Damage but no kill] = β ,   P[Kill] = γ ,

$\alpha + \beta + \gamma = 1$

$P[x < D < x + dx \,|\, damage] = f_D(x)dx$, $x \leq b$ $\overset{\text{(maximum tolerable}}{\text{damage)}}$

n.b. target is destroyed either by killing or absorbing

damage $\geq b$

cf of $f_D(x) = \psi_D(u)$

$D(t)$ = rv total damage in $(0, t)$ | target still alive

$F_{D(t)}(x)$ = $P[D(t) < x$ | target alive$]$

cf of $F_{D(t)}(x)$ w r t x = $\Psi_{D(t)}(u)$

$$w(t) = \int_0^t \sum_{n=0}^\infty p_N(n ; t) \, r_n(\xi) d\xi$$

$$\Psi_{D(t)}(u) = - \frac{e^{[\beta \Psi_D(u) - 1 + \alpha]w(t)}}{iu}$$

$$F_{D(t)}(x) = \frac{1}{2\pi i} \int_{-\infty + i\epsilon}^{\infty + i\epsilon} e^{[-iux + (\beta \Psi_D(u) + \alpha - 1)]w(t)} \frac{du}{u}$$

T_K = rv time to a kill; $F_{T_K}^c(t)$ = $P[T_K > t]$

$$\frac{\partial}{\partial t} F_{T_K}^c(t) = \gamma \sum_{n=0}^\infty p_N(n ; t) \, r_n(t)$$

$$h(t) = \gamma \sum_{n=0}^\infty p_N(n ; t) \, r_n(t) \, F_{D(t)}(b) - F_{T_K}(t) \frac{\partial}{\partial t} F_{D(t)}(b) \qquad \text{N \& J1}$$

III. ROUND-DEPENDENT HIT PROBABILITIES

A. UNLIMITED AMMUNITION

p_n = P[Hit on n-th round | n-th round fired]

$p_N(n)$ = ρ_n = P[Hit on n-th round , n-th round fired]

$$\rho_n = p_n \prod_{j=0}^{n-1} (1 - p_j) = p_n \prod_{j=0}^{n-1} q_j \quad \text{where} \quad q_j = 1 - p_j, \quad q_0 = 1$$

$$\Phi(u) = \sum_{n=1}^{\infty} \rho_n \phi^n(u)$$

Bh5,
A4
$$= \phi(u) - [1 - \phi(u)] \sum_{n=1}^{\infty} \phi^n(u) \prod_{j=0}^{n} q_j .$$

W2   or $\Phi(u) = G_N(\phi(u))$

*
$$\Psi(u,y) = \left[U(y) + \sum_{n=1}^{\infty} \prod_{j=0}^{n} q_j \phi^n(u) \right] e^{iuy - \int_0^y \lambda(\xi)d\xi}$$

Example 1:

$$q_j = \left(\frac{N}{j} - 1 \right) a = \frac{N - j}{j} a; \quad j = 1, 2, \dots, N$$
$$\left. \begin{array}{l} \\ = 0 \quad ; \quad j = N+1, N+2, \dots \end{array} \right\} \quad N, a \text{ constants}$$

$$a \le \frac{1}{N - 1} ; \quad N \in I^+; \quad a > 0$$

$$\Phi(u) = 1 - [1 - \phi(u)][1 + a\phi(u)]^{N-1}$$
and if $X \sim \text{ned}(r)$

$$\Phi(u) = 1 + \frac{iu}{(r - iu)^N} [(1+a)r - iu]^{N-1}$$

O

Example 2:

$$q_j = \frac{q_1}{j} \; ; \quad j = 1, 2, \ldots, ; \quad 1 - q_1 = \text{1-st round hit probability}$$

$$\Phi(u) = 1 - [1 - \phi(u)] \exp [q_1 \, \phi(u)]$$

& if $X \sim \text{Erlang}(2, r)$

$$\Phi(u) = \frac{(r - iu)^2 + u(u + 2ir) \exp\left[\dfrac{q_1 r^2}{(r - iu)^2}\right]}{(r - iu)^2}$$

Example 3:

Same as Example 2, except, let $X \sim \text{ned}(r)$

$$\Phi(u) = \frac{r - iu\left[1 - \exp\left(\dfrac{q_1 r}{r - iu}\right)\right]}{r - iu}$$

A4

Example 4:

$$P_N(n) = \rho_n = \binom{n + k - 1}{k} \xi^{k+1}(1 - \xi)^{n-1} \; , \begin{cases} n = 1, 2, \ldots, \\[6pt] 0 < \xi < 1, \; k \geq 0 \end{cases}$$

$$\Rightarrow P_1 = \xi^{k+1}$$

$$\vdots$$

$$P_n = \frac{\xi}{\displaystyle\sum_{j=0}^{k} \frac{k(k+1) \cdots (k - j + 1)}{n(n+1) \cdots (n + j - 1)} \left(\frac{1 - \xi}{\xi}\right)^j}$$

$$\left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} \begin{array}{l} \text{monotone} \\ \text{non-decreasing} \\ \text{sequence} \\ \text{for } k \geq 1 \end{array}$$

$$\vdots$$

$$P_\infty = \xi$$

Define

$$p \triangleq \frac{1}{\sum_{n=1}^{k} n p_N(n)} = \frac{1}{\mu_N} = \frac{\xi}{1 + k(1 - \xi)}$$

$$p_1 < p < p_\infty$$

$$\lim_{k \to 0} \ p_N(n) \longrightarrow \text{Geometric } (\xi = p)$$

$$\lim_{k \to \infty} \ p_N(n) \longrightarrow \text{Poisson } (\lambda = q/p)$$

$$p_1 = e^{-q/p}$$

.
.
.

$$p_n = \frac{1}{1 + \frac{q}{pn} + \frac{q^2}{p^2(n)(n + 1)}} + \cdots$$

.
.
.

$$p_\infty$$

MLE $\quad \hat{p} = 1/\bar{n} \quad$ where $\quad \bar{n} = \dfrac{n_1 + n_2 + \cdots + n_\ell}{\ell}, \quad$ where

$n_i = $ i-th sample from $p_N(n)$ where there are ℓ samples

$\hat{k}$ is the solution to

$$\sum_{j=1}^{\ell} \left(\frac{1}{\hat{k} + 1} + \cdots + \frac{1}{\hat{k} + n_j - 1} \right) - \ell \ \ln \frac{\hat{k} + \bar{n}}{\hat{k} + 1} = 0$$

C33

$$\Phi(u) = \phi(u)\left[\frac{\xi}{1 - (1 - \xi)\phi(u)}\right]^{k+1}$$

B. LIMITED AMMUNITION

1. Fixed Supply of k Rounds

$$\Phi_0(u) = \prod_{n=0}^{k} q_n \phi^k(u)$$

$$\psi(u,y) = \left[U(y) + \sum_{n=1}^{k-1} \prod_{j=0}^{n} q_j \phi^n(u)\right] e^{iuy - \int_0^y \lambda(\xi)d\xi} , \quad \text{where}$$

the second term in the brackets is zero for $R = 1$

$$\Phi_1(u) = \sum_{n=1}^{k} \rho_n \phi^n(u) = \sum_{n=1}^{k} p_n \prod_{j=0}^{n-1} q_j \phi^n(u)$$

$$h(t) = h_1(t) + \prod_{n=0}^{k} q_n \delta(t - \infty)$$

2. Ammunition Supply a RV

$$P[I = i] = \alpha_i ; \quad \sum_{i=1}^{\infty} \alpha_i = 1$$

$$\Phi_0(u) = \sum_{i=0}^{\infty} \alpha_i \prod_{n=0}^{i} q_n \phi^i(u)$$

*
$$\psi(u,y) = \left[U(y) + \sum_{i=0}^{\infty} \alpha_i \sum_{n=1}^{i-1} \prod_{j=0}^{n} q_j \, \phi^n(u) \right] e^{iuy - \int_0^y \lambda(\xi)d\xi}$$

where 2-nd term in brackets is zero for $i = 0,1$.

$$\Phi_1(u) = \sum_{i=0}^{\infty} \alpha_i \sum_{n=1}^{i} \rho_n \, \phi^n(u)$$

Bh5
$$h(t) = h_1(t) + \sum_{i=0}^{\infty} \alpha_i \prod_{n=1}^{i} q_n \, \delta(t - \infty)$$

IV. TIME-DEPENDENT HIT PROBABILITIES

A. p A FUNCTION OF TIME SINCE START

$p(t) = P[H \mid a \text{ firing at time } t]$

$q(t) = 1 - p(t)$

cf of $q(t) = \Omega(u)$

$$h^0(t) = q(t) f(t) + q(t) \int_0^t h^0(t - \tau) f(\tau)d\tau$$

$$h(t) = f(t) + \int_0^t h^0(t - \omega) f(\omega)d\omega - h^0(t)$$

$$\Theta_0(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Omega(w) \, \phi(u - w)dw$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \Omega(w) \, \phi(u - w) \, \Theta_0(u - w)dw$$

$$\Phi(u) = \phi(u) + [\phi(u) - 1] \Theta_0(u)$$

If $\Omega(w)$ has one (not necessarily simple) pole at $-w_0$ in the lower half of the complex w plane, then

$$S(u,w_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Omega(w) \phi(u - w) dw \qquad \text{(known)}$$

and

$$\Theta_0(u) = S(u,w_0) + S(u,w_0) \Theta_0(u + w_0)$$

or

$$\Theta_0(u) = \sum_{j=0}^{\infty} \prod_{k=0}^{j} S(u + kw_0, w_0) .$$

<u>Example 1</u>: $X = ned(r)$, $q(t) = \eta e^{-\rho t}$, $\Omega(u) = \eta/(\rho - iu)$, then

$$S(u, w_0) = \frac{\eta r}{r + \rho - iu}$$

$$\Theta_0(u) = \eta r e^{\eta r/\rho} \sum_{k=0}^{\infty} \frac{i(-1)^k (\eta r/\rho)^k}{k! \, [u + i(r + (k + 1)\rho)]}$$

$$h^0(t) = r e^{-rt} \eta e^{-\rho t} e^{-\eta r/\rho[e^{-\rho t}-1]}$$

$$h(t) = r e^{-rt} e^{-\eta r/\rho[e^{-\rho t}-1]} (1 - \eta e^{-\rho t}) . \qquad \text{A7}$$

C36

Example 2: ned(r) only:

$p = p(t)$ - continuous, integrable; $0 \leq p \leq 1$

$$\lim_{a \to \infty} \int_0^a p(x)dx \longrightarrow \infty$$

$$h(t) = rp(t)\, e^{-r \int_0^t p(x)dx}$$

Sub-Example: A Closing Engagement

$$p(t) = \frac{a}{(r_s - vt)^2} \quad ; \quad 0 \leq t \leq t_0 \left.\begin{array}{c}\\\\\end{array}\right\} \quad a, r_s, v, t_0 \text{ positive constants}$$

$$= \frac{a}{(r_s - vt_0)^2} \quad ; \quad t \geq t_0 \quad a \leq (r_s - vt_0)^2, \; vt_0 < r_s$$

$$\int_0^t p(x)dx = \frac{at}{r_s(r_s - vt)} \quad ; \quad 0 \leq t \leq t_0$$

$$= \frac{a(r_s t - vt_0^2)}{r_s(r_s - vt_0)^2} \quad ; \quad t \geq t_0 \quad .$$

Example 3: p as in Example 2.

$r(t)\Delta t + O(\Delta t) = P[\text{Exactly } 1 \text{ round fired in } (t, t + \Delta t)]$

which means firing is a non-stationary Poisson process,

T1 $\qquad h(t) = r(t)\, p(t)\, e^{-\int_0^t r(x)p(x)dx}$.

B. p A FUNCTION OF TIME SINCE LAST FIRING

$p(x) = P[H \mid$ firing at IFT $x]$, i.e., hit probabilities are a function
of IFT

$q(x) = 1 - p(x)$

cf $q(x) = \Omega_X(u)$.

Limited Ammunition Fixed at k Rounds

$$\Phi_0(u) = \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(u - w) \, \Omega_X(w) dw \right]^k \triangleq [S(u)]^k$$

$$\Psi(u,y) = \left[U(y) + \frac{S(u) - \Phi_0(u)}{1 - S(u)} \right] e^{iuy - \int_0^y \lambda(\xi) d\xi}$$

$$\Phi_1(u) = \left[\phi(u) - S(u) \right] \left[\frac{1 - \Phi_0(u)}{1 - S(u)} \right]$$

$$h(t) = h_1(t) + \delta(t - \infty) \left[\int_0^{\infty} f_X(x) \, q(x) dx \right]^k$$

Example: $X \sim ned(r)$; unlimited ammunition

$q(x) = e^{-\sigma t}$

$$\Phi(u) = \frac{r\rho}{(r - iu)(\rho - iu)} \quad .$$

Bh⊥

V. LIMITED AMMUNITION

A. AMMUNITION SUPPLY A RV

$$P[I = i] = \alpha_i; \quad P[I = \infty] = \alpha_\infty; \quad \alpha_\infty + \sum_{i=0}^{\infty} \alpha_i = 1$$

$$h(t) = \alpha_\infty \, p \sum_{j=0}^{\infty} q^j \, f^{(j+1)*}(t) + p \sum_{i=1}^{\infty} \alpha_i \sum_{j=0}^{i-1} q^j \, f^{(j+1)*}(t)$$

$$+ \, \delta(t - \infty) \sum_{i=0}^{\infty} \alpha_i \, q^i$$

$$= h_1(t) + \delta(t - \infty) \sum_{i=0}^{\infty} \alpha_i \, q^i$$

$$H(t) = \int_0^t h(\xi) d\xi = H_1(t)$$

$$H^c(t) \int_t^{\infty} h(\xi) d\xi = H_1^c(t) + \sum_{i=0}^{\infty} \alpha_i \, q^i$$

A1,
A & G1
$$\Phi_1(u) = \frac{p\phi(u)}{1 - q\phi(u)} \left\{ 1 - \sum_{i=0}^{\infty} \alpha_i [q\phi(u)]^i \right\}$$

$$P(H) = 1 - \sum_{i=0}^{\infty} \alpha_i \, q^i$$

$$P(\bar{H}) = \sum_{i=0}^{\infty} \alpha_i q^i$$

$$P(N = n \mid H) = \frac{1}{P(H)} pq^{n-1} \left(\alpha_\infty + \sum_{i=n}^{\infty} \alpha_i \right), \qquad n \geq 1$$

$$E(N \mid H) = \frac{1}{P(H)} \left[\frac{1 - \sum_{i=0}^{\infty} \alpha_i q^i}{p} - \sum_{i=1}^{\infty} i\alpha_i q^i \right]$$

$$E(N^2 \mid H) = \frac{1}{P(H)} \left[\frac{1+q}{p^2} \left(1 - \sum_{i=0}^{\infty} \alpha_i q^i \right) - \frac{2}{p} \sum_{i=1}^{\infty} i\alpha_i q^i \right.$$
$$\left. - \sum_{i=1}^{\infty} i^2 \alpha_i q^i \right]$$

$$P(N \geq n_0 \mid H) = \frac{1}{P(H)} \left[\alpha_\infty + \sum_{i=0}^{\infty} \alpha_{n_0+i}(1 - q^{i+1}) \right] q^{n_0-1}$$

$$P(N = n \mid \bar{H}) = \frac{1}{P(\bar{H})} \alpha_n q^n, \qquad\qquad n \geq 0$$

$$P(N = n) = \alpha_0, \qquad\qquad n = 0$$

$$= pq^{n-1} \left(\alpha_\infty + \sum_{i=n}^{\infty} \alpha_i \right) + \alpha_n q^n, \qquad n \geq 1 \ .$$

B. **AMMUNITION SUPPLY FIXED**

$$\alpha_k = 1; \quad \alpha_\infty = \alpha_i = 0; \quad i \neq k$$

A1

$$\Phi_1(u) = \frac{p\phi(u)}{1 - q\phi(u)} \left[1 - (q\phi(u))^k \right]$$

$$P(H) = 1 - q^k = 1 - P(\bar{H})$$

$$P(N = n \mid H) = \frac{pq^{n-1}}{P(H)} \qquad\qquad n \leq k$$

$$E(N \mid H) = \frac{1}{P(H)} \left[\frac{1 - q^k}{p} - kq^k \right]$$

$$E(N^2 \mid H) = \frac{1}{P(H)} \left[\frac{1 + q}{p^2} (1 - q^k) - \frac{2}{p} kq^k - k^2 q^k \right]$$

$$P(N \geq n_0 \mid H) = \frac{1}{P(H)} \left(q^{n_0 - 1} - q^k \right)$$

$$P(N = n \mid \bar{H}) = 0 , \qquad\qquad n \neq k$$

$$= \frac{q^k}{P(\bar{H})} , \qquad\qquad n = k$$

$$P(N = n) = 0 \qquad , \qquad\qquad n = 0, n > k$$

$$= pq^{n-1} , \qquad\qquad 1 \leq n < k$$

A & G1

$$= q^{k-1} , \qquad\qquad n = k .$$

Example 1:

$$\alpha_k = 1; \quad \alpha_\infty = \alpha_i = 0; \quad i \neq k$$

$$X \sim \text{ned}(r)$$

$$\phi_1(u) = \frac{pr}{pr - iu} \left[1 - \left(\frac{qr}{r - iu} \right)^k \right] . \tag{A1}$$

Example 2:

$$\alpha_i = (1 - \alpha_\infty)(1 - \alpha)\alpha^i; \quad \alpha_\infty \neq 0; \quad 0 < \alpha < 1$$

$$X \sim \text{ned}(r)$$

$$\phi_1(u) = \frac{pr}{pr - iu} \left\{ \frac{-r\alpha q + (r - iu)[\alpha_\infty(1 - \alpha) + \alpha]}{r(1 - \alpha q) - iu} \right\} \tag{A1, A \& G1}$$

$$P(H) = 1 - \frac{(1 - \alpha_\infty)(1 - \alpha)}{1 - \alpha q} = 1 - P(\overline{H})$$

$$P(N = n \mid H) = \frac{pq^{n-1}}{P(H)} [\alpha_\infty + (1 - \alpha_\infty)\alpha^n], \qquad n \geq 1$$

$$E(N \mid H) = \frac{1}{P(H)(1 - \alpha q)} \left[\frac{\alpha p + (1 - \alpha)\alpha_\infty}{p} - \frac{(1 - \alpha)(1 - \alpha_\infty)\alpha q}{1 - \alpha q} \right]$$

$$E(N^2 \mid H) = \frac{1}{P(H)} \left[\frac{1 + q}{p} \left(1 - \frac{(1 - \alpha_\infty)(1 - \alpha)}{1 - \alpha q} \right) - \frac{2(1 - \alpha)(1 - \alpha_\infty)\alpha q}{p(1 - \alpha q)^2} \right.$$

$$\left. - \frac{\alpha q(1 - \alpha)(1 - \alpha_\infty)(1 + \alpha q)}{(1 - \alpha q)^3} \right]$$

$$P(N \geq n_0 \mid H) = \frac{1}{P(H)} \left[\alpha_\infty + (1 - \alpha_\infty)\alpha^{n_0} \left(\frac{p}{1 - \alpha q} \right) \right] q^{n_0 - 1}$$

$$P(N = n \mid \bar{H}) = \frac{(1 - \alpha)(1 - \alpha_\infty)(\alpha q)^n}{P(\bar{H})} \quad , \qquad n \geq 0$$

$$P(N = n) = \alpha_0 \qquad , \quad n = 0$$

A & G1
$$= pq^{n-1}[\alpha_\infty + (1 - \alpha_\infty)\alpha^n] + (1-\alpha_\infty)(1-\alpha)(\alpha q)^n, \quad n \geq 1 \quad .$$

Example 3:

$$\alpha_\infty = 0$$

$$\alpha_i = \left(\frac{1}{1 + \alpha} \right)^k \binom{k}{i} \alpha^i; \quad i = 0,1,2,\ldots,k, \quad \alpha > 0$$

$$= 0 \qquad ; \quad i = k + 1, \ k + 2, \ldots$$

$$X \sim \text{ned}(r)$$

A1,
A & G1
$$\Phi_1(u) = \frac{pr}{pr - iu} \left\{ 1 - \left(\frac{1}{1 + \alpha} \right)^k \left(1 + \frac{\alpha q r}{r - iu} \right)^k \right\} \quad .$$

Example 4:

$$\alpha_\infty = 0; \quad \alpha_i = \frac{e^{-\alpha} \alpha^i}{i!}; \quad i = 1,2,\ldots, \quad \alpha > 0$$

$$X \sim \text{ned}(r)$$

$$\Phi_1(u) = \frac{pr \left[1 - \exp \left[-\alpha \left(\frac{pr - iu}{r - iu} \right) \right] \right]}{pr - iu} \quad .$$

Example 5:

$$\alpha_\infty = 0; \qquad \alpha_i = (1 - \alpha)\alpha^i; \qquad i = 1,2,\ldots,; \qquad 0 < \alpha < 1$$

$$X \sim \text{Erlang}(2,r)$$

$$\Phi_1(u) = \frac{\alpha p r^2}{(r - iu)^2 - \alpha q r^2} \quad . \qquad\qquad\qquad\qquad \text{A1}$$

VI. RELIABILITY

Weapons fail on firing. The preceding section on limited ammunition can also be interpreted as a reliability situation.

A. CONSTANT PROBABILITY OF FAILURE ON EACH ROUND

p = probability of a hit

q = probability of a miss

v = probability of a failure (i.e., on each round fired there is a probability of a failure). Firing ceases at a failure.

$$p + q + v = 1$$

$$h_1(t) = \sum_{i=1}^{\infty} pq^{i-1} r^{i*}(t)$$

$$h_0(t) = \sum_{i=1}^{\infty} vq^{i-1} r^{i*}(t) \qquad\qquad\qquad\qquad \text{T1}$$

$$\Phi(u) = \Phi_1(u) + \Phi_0(u), \quad \text{where}$$

$$\Phi_1(u) = \frac{p\phi(u)}{1 - q\phi(u)} \qquad\qquad\qquad\qquad\qquad *$$

$$* \qquad \Phi_0(u) = \frac{v\phi(u)}{1 - q\phi(u)} \quad .$$

Example 1:

$X \sim \text{ned}(r)$

$$h_1(t) = pre^{-(p+v)rt}$$

$$h_0(t) = vre^{-(p+v)rt}$$

T1 $\qquad h(t) = (p + v)re^{-(p+v)rt} \quad .$

B. ROUND NUMBER ON WHICH FAILURE OCCURS IS A DISCRETE RV

Let $K = $ RV, round number on which a failure occurs (it is discovered one attempted firing later).

$P[K = k + 1] = \alpha_k = P[\text{A failure to fire occurs on round } k+1]$

$$\sum_{k=0}^{\infty} \alpha_k = 1$$

$$h_1(t) = \sum_{i=1}^{\infty} pq^{i-1} f^{i*}(t) \left(\sum_{k=1}^{\infty} \alpha_k \right)$$

T2 $\qquad h_0(t) = \sum_{i=0}^{\infty} \alpha_i q^{i+1} f^{(i+1)*}(t)$

A6 $\qquad \Phi_1(u) = \frac{p\phi(u)}{1 - q\phi(u)} \left[1 - \sum_{k=0}^{\infty} \alpha_k q^k \phi^k(u) \right]$

$$* \qquad \Phi_0(u) = \sum_{i=0}^{\infty} \alpha_i q^{i+1} \phi^{i+1}(u)$$

$$G_K(z) = \sum_{i=0}^{\infty} \alpha_i z^{i+1} \quad (z \text{ transform of } K)$$

$$\Phi_1(u) = \frac{p\phi(u)}{1 - q\phi(u)} \left[1 - \frac{G_K[q\phi(u)]}{q\phi(u)} \right] \qquad \text{A6}$$

$$\Phi_0(u) = G_K[q\phi(u)] \quad . \qquad *$$

VII. LIMITED TIME-DURATION

A. T_L A RV

cf of $f_{T_L}(t) = \Theta(w)$

$$h_1(t) = P(H) h_H(t) = \frac{1}{2\pi} \left[\int_{-\infty}^{\cdot} e^{-iut} \Phi(u) du \right] F_{T_L}^c(t)$$

$$\Phi_1(u) = P(H) \Psi_H(u) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Phi(u - w)[\Theta(w) - 1]dw}{w}$$

$$= \frac{1}{2\pi i} \int_L \frac{\Phi(u - w) \Theta(w)dw}{w} \qquad *$$

$$P(H) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Phi(-u)[\Theta(u) - 1]du}{u} = \frac{1}{2\pi i} \int_L \frac{\Phi(-u) \Theta(u)du}{u}$$

$$p(H) \mu_n(H) = \frac{1}{2\pi i^{n+1}} \int_{-\infty}^{\infty} \frac{\overset{(n)}{\cdot}(-w)[\Theta(w) - 1]dw}{w}$$

$$= \frac{1}{2\pi i^{n+1}} \int_{-} \frac{\Phi^{(n)}(-w) \Theta(w)dw}{w}$$

$$\phi^{(n)}(-w) = \frac{\partial}{\partial u^n} \Phi(u - w)\big|_{u=0}$$

$$h_0(u) = P(\overline{H}) \, h_{\overline{H}}(t) = \frac{f_{T_L}(t)}{2} \int_{-\infty}^{\infty} \frac{e^{-iut}[\Phi(u) - 1]du}{u}$$

$$= \frac{f_{T_L}(t)}{2\pi i} \int_L \frac{e^{-iut} \Phi(u)du}{u}$$

$$* \quad \Phi_0(u) = P(\overline{H}) \, \psi_{\overline{H}}(u) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Theta(u - w)[\Phi(w) - 1]dw}{w}$$

$$= \frac{1}{2\pi i} \int_L \frac{\Theta(u - w) \, \Phi(w)dw}{w}$$

$$* \quad P(\overline{H}) = 1 - P(H) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Theta(-u)[\Phi(u) - 1]du}{u} = \frac{1}{2\pi i} \int_L \frac{\Theta(-u) \, \Phi(u)du}{u}$$

$$P(\overline{H}) \, \mu_n(\overline{H}) = \frac{1}{2\pi i^{n+1}} \int_{-\infty}^{\infty} \frac{\Theta^{(n)}(-w)[\Phi(w) - 1]dw}{w}$$

$$= \frac{1}{2\pi i^{n+1}} \int_L \frac{\Theta^{(n)}(-w) \, \Phi(w)dw}{w}$$

$$\Theta^{(n)}(w) = \frac{\partial^n \Theta(u - w)}{\partial u^n} \Big|_{u=0}$$

$$h(t) = h_H(t) \, P(H) + h_{\overline{H}}(t) \, p(\overline{H}) \quad .$$

Example: Let $X = \text{Erlang}(2, r)$

$$f_{T_L}(t) = \frac{1}{\tau} e^{-t/\tau}$$

$$P(H) = \frac{4pr^2 \tau^2}{4pr^2 \tau^2 + 4r\tau + 1}$$

$$P(H) \, \mu_1(H) = \frac{8pr^2(2r + 1/\tau)}{[(2r + 1/\tau)^2 - 4r^2 q]^2}$$

$$\mu_1(H) = \frac{2\tau[2r\tau + 1]}{4pr^2 \tau^2 + 4r\tau + 1}$$

$$h_1(t) = P(H) h_H(t) = \frac{2pr}{\sqrt{q}} e^{-(2r+1/\tau)t} \sinh 2r \sqrt{q} \, t \quad .$$

B. $\underline{T_L \text{ A CONSTANT } (\tau)}$

$$h_1(t) = P(H) h_H(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iut} \Phi(u)du \; ; \qquad\qquad t < \tau$$

$$\Phi_1(u) = P(H) \psi_H(u) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Phi(u-w)[e^{iw\tau} - 1]dw}{w}$$

$$P(H) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Phi(-u)[e^{iu\tau} - 1]du}{u}$$

$$P(H) \mu_n(H) = \frac{1}{2\pi i^{n+1}} \int_{-\infty}^{\infty} \frac{\Phi^{(n)}(-w)[e^{iw\tau} - 1]dw}{w}$$

$$h_0(u) = P(\bar{H}) h_{\bar{H}}(t) = \frac{\delta(t-\tau)}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iu\tau}[\Phi(u) - 1]du}{u}$$

$$= \frac{\delta(t-\tau)}{2\pi} \int_L \frac{e^{-iu\tau} \Phi(u)du}{u}$$

$$\Phi_0(u) = P(\bar{H})\psi_{\bar{H}}(u) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i(u-w)\tau} [\Phi(u) - 1]dw}{w}$$

$$= \frac{1}{2\pi i} \int_L \frac{e^{i(u-w)\tau} \Phi(w)dw}{w}$$

$$P(\bar{H}) = 1 - P(H) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-iu\tau}[\Phi(u) - 1]du}{u} = \frac{1}{2\pi i} \int_L \frac{e^{-iu\tau} \Phi(u)du}{u}$$

$$P(\bar{H}) \mu_n(\bar{H}) = \frac{\tau^n}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-iw\tau}[\Phi(w) - 1]dw}{w} = \frac{\tau^n}{2\pi i} \int_L \frac{e^{-iw\tau} \Phi(w)dw}{w}$$

A & G2 $h(t) = h_H(t) P(H) + h_{\bar{H}}(t) P(\bar{H})$.

VIII. INTERRUPTED FIRING

Firing with Weapons Which Fail and Can be Repaired (Replaced)

Marksman fires until he hits or weapon fails. Time-to-failure is an independent rv. Time-to-repair or replace is an independent rv. Process continues until marksman hits.

A. LIMITED AMMUNITION - FIXED AT k ROUNDS

rv time-to-failure $T_L \sim ned(r_L)$

rv time-to-repair (replace), no contact $= T_{\bar{C}}$ with cf $\Theta_{\bar{C}}(u)$

$y_L =$ time since last failure

$$\Phi_O(u) = \left[\frac{q(r_L - iu)\,\phi(u + ir_L)}{(r_L - iu) - r_L[1 - \phi(u + ir_L)]\Theta_{\bar{C}}(u)} \right]^k \quad \begin{array}{l}\text{cf of Improper}\\ \text{pdf, time-to-a-}\\ \text{draw}\end{array}$$

$$\Psi(u,y) = [U(y) + \Phi_O(u)]\, e^{(iu - r_L)y - \int_0^y \lambda(\xi)d\xi}$$

$$\psi(u,y_L) = \frac{r_L(r_L - iu)[1 - \Theta_O(u)]\,\phi(u + ir_L)\, e^{-iuy_L - \int_0^{y_L}\lambda(\xi)d\xi}}{(r_L - iu)[1 - q\phi(u + ir_L)] - r_L(1 - \phi(u + ir_L))\,\Theta_{\bar{C}}(u)}$$

where

$$\Theta_O(u) = \frac{(r_L - iu)[q\phi(u + ir_L) - \Phi_O(u)] + r_L[1 - \phi(u + ir_L)]\,\Theta_{\bar{C}}(u)}{(r_L - iu)[1 - q\phi(u + ir_L)] - r_L[1 - \phi(u + ir_L)]\,\Theta_{\bar{C}}(u)}$$

$$\Phi(u) = \frac{p(r_L - iu)\,\phi(u + ir_L)}{(r_L - iu)[1 - q\phi(u + ir_L)] - r_L[1 - \phi(u + ir_L)]\,\Theta_{\bar{C}}(u)}$$

$$\cdot \left\{ 1 - \left[\frac{(r_L - iu)\,\phi(u + ir_L)\,q}{(r_L - iu) - r_L[1 - \phi(u + ir_L)]\,\Theta_{\bar{C}}(u)} \right]^k \right\} + q^k$$

B.   **AMMUNITION LIMITATION A   RV**

$$P[I = i] = \alpha_i \quad ; \qquad \sum_{i=0}^{\infty} \alpha_i = 1$$

$$\Phi_0(u) = \sum_{i=0}^{\infty} \alpha_i \left[\frac{q(r_L - iu)\,\phi(u + ir_L)}{(r_L - iu) - r_L[1 - \phi(u + ir_L)]\Theta_{\overline{C}}(u)} \right]^i$$

$$\Phi(u) = \frac{p(r_L - iu)\,\phi(u + ir_L)}{(r_L - iu)[1 - q\phi(u + ir_L)] - r_L[1 - \phi(u + ir_L)]\Theta_{\overline{C}}(u)}$$

$$\cdot \left\{ 1 - \sum_{i=0}^{\infty} \alpha_i \left[\frac{(r_L - iu)\,\phi(u + ir_L)q}{(r_L - iu) - r_L[1 - \phi(u + ir_L)]\Theta_{\overline{C}}(u)} \right]^i \right\}$$

$$+ \sum_{i=0}^{\infty} \alpha_j\, q^i \; .$$

C.   **UNLIMITED AMMUNITION**

$$\Phi(u) = \frac{p(r_L - iu)\,\phi(u + ir_L)}{(r_L - iu)[1 - q\phi(u + ir_L)] - r_L[1 - \phi(u + ir_L)]\Theta_{\overline{C}}(u)} \; .$$

Example:

Unlimited ammunition; $X \sim \text{ned}(r)$; $T_{\overline{C}} \sim \text{ned}(r_{\overline{C}})$

$$\Phi(u) = \frac{pr(iu - r_{\overline{C}})}{u^2 + iu(r_{\overline{C}} + r_L + pr) - prr_{\overline{C}}} \; .$$

Bh3(4)

D. **UNLIMITED AMMUNITION, EXCEPT FIRING CONTINUES FOR AN UNLIMITED NUMBER OF HITS**

I.e., firing never ceases and the number of hits is counted.

$X \sim \text{ned}(r)$ (derived in a naval warfare context -- submarine versus anti-submarine vessels)

Time is zero at initiation of first contact (combat), and

X_C - rv time in contact $\sim \text{ned}(r_C)$

$\phi_{\bar{C}}(u)$ - cf of $X_{\bar{C}}$, rv time not in contact

R - rv number of hits

T_R - rv time to R hits

$T_{R,\bar{C}}$ - rv time to R hits, not in contact

$T_{R,C}$ - rv time to R hits, in contact

$\Theta_{R,\bar{C}}(u)$ - cf of DF of $T_{R,\bar{C}}$

$\Theta_{R,C}(u)$ - cf of DF of $T_{R,C}$

$\Theta_R(u)$ - cf of DF of $T_R = \Theta_{R,\bar{C}}(u) + \Theta_{R,C}(u)$

T_C - rv time since start, in contact

$T_{\bar{C}}$ - rv time since start, not in contact

$\Theta_C(u)$ - cf of DF of T_C

$\Theta_{\bar{C}}(u)$ - cf of DF of $T_{\bar{C}}$

$$G_{R,C}^{(z)} = \frac{1}{(-iu + r_C + pr - r_C \phi_{\overline{C}}(u) - prz)}$$

$$G_{R,\overline{C}}(z) = \frac{r_C[1 - \phi_{\overline{C}}(u)]}{-iu[-iu + r_C + pr - r_C \phi_{\overline{C}}(u) - prz]} \quad .$$

Expanding in powers of z, the coefficient of z^i is $\Theta_{i,C}(u)$ and $\Theta_{i,\overline{C}}(u)$, respectively; also,

$$G_{R,C}(1) = cf \text{ of } P[\text{Being in contact at time } t] = \Theta_C(u) \qquad *$$

$$G_{R,\overline{C}}(1) = cf \text{ of } P[\text{Not being in contact at time } t] = \Theta_{\overline{C}}(u) \qquad *$$

$$\Psi_{R,\overline{C}}(u,y_C) = \Psi_{R,\overline{C}}(u,0)e^{iuy_C - \int_0^{y_C} \lambda_{\overline{C}}(\xi)d\xi}$$

where y_C is time since last contact

$$\Psi_{R,\overline{C}}(u,0) = r_C \Theta_{R,C}(u)$$

$$G_{\Psi_{\overline{C}}}(z) = r_C G_{R,C}^{(z)} \quad .$$

Example:

$$X_{\overline{C}} \sim ned(r_{\overline{C}})$$

$$G_{R,C}^{(z)} = \frac{r_{\overline{C}} - iu}{-u^2 - iu(r_{\overline{C}} + r_C + pr - prz) + r_{\overline{C}} pr(1 - z)}$$

$$G_{R,\overline{C}}(z) = \frac{r_C}{-u^2 - iu(r_{\overline{C}} + r_C + pr - prz) + r_{\overline{C}}\, pr(1 - z)}$$

$$\Theta_{R,C}(u) = \frac{(pr)^R\ (r_{\overline{C}} - iu)^{R+1}}{[-u^2 - iu(r_{\overline{C}} + r_C + pr) + pr\,r_{\overline{C}}]^{R+1}}$$

$$\Theta_{R,\overline{C}}(u) = \frac{r_C(pr)^R\ (r_{\overline{C}} - iu)^R}{[-u^2 - iu(r_{\overline{C}} + r_C + pr) + pr\,r_{\overline{C}}]^{R+1}}$$

$$\Theta_R(u) = \Theta_{R,\overline{C}}(u) + \Theta_{R,C}(u)$$

M&A1 $$E[T_R] = R\ \frac{r_{\overline{C}} + r_C}{p\,r\,r_{\overline{C}}}\ ,$$

n.b., since firing is unlimited, this contains time to R

hits when $R + 1$ hits have been made, etc.

IX. TIME-OF-FLIGHT INCLUDED

$X = rv$ (CRIFT)

T_F $= rv$ time-of-flight (TOF)

T_K $= rv$ time-to-fire killing round

T_M $= rv$ time-to-hit target

f_{T_F} $= pdf$ of $T_F \sim cf = \phi_F(u)$

h_K $= pdf$ of $T_K \sim cf = \Phi_K(u)$

$$h(t) = \text{pdf of } T_M \qquad cf = \Phi(u)$$

$$f(t) = \text{pdf of } X \sim cf = \phi(u)$$

A. <u>NO DELAY BETWEEN ROUNDS FIRED</u>

Marksman fires as rapidly as possible; then $T_K = \Sigma X_i, X_i$ are iid $\sim X$, $i = 1,2,\ldots,$ till a killing round is fired. $T_M = T_K + T_F$.

$$\Phi_K(u) = \frac{p\phi(u)}{1 - q\phi(u)}$$

$$\Phi(u) = \frac{p\phi(u) \ \phi_F(u)}{1 - q\phi(u)}$$

$$\Phi(u) = \frac{p\phi(u) \ e^{iu\tau}}{1 - q\phi(u)} \ ; \quad \text{when} \quad T_F = \tau, \quad \text{a constant.}$$

<u>Example 1</u>: Let $X \sim \text{ned}(r)$,

$$f_{T_F}(t) = \frac{1}{\tau} e^{-t/\tau}$$

$$h(t) = \frac{pr(e^{-prt} - e^{-t/\tau})}{1 - pr\tau} \ ; \qquad t > 0, \qquad pr \neq \frac{1}{\tau}$$

$$= (pr)^2 \ te^{-prt} \qquad ; \qquad t > 0, \qquad pr = \frac{1}{\tau} \qquad *$$

$$\phi(u) = \frac{pr}{(pr - iu)(1 - iu\tau)} \ .$$

<u>Example 2</u>: let $X \sim \text{ned}(r); \ T_F = \tau,$ a constant.

$$h(t) = pre^{-pr(t-\tau)}, \qquad t \geq \tau$$

$$= 0 \qquad , \qquad t < \tau$$

$$\Phi(u) = \frac{pre^{iu\tau}}{pr - iu} \qquad .$$

B. DELAY BETWEEN ROUNDS FIRED

Each round fired is allowed to land first and then the process starts over again.

$$T_M = (X_1 + T_{F_1}) + (X_2 + T_{F_2}) + \cdots + (\text{Until target is hit})$$

$$\Phi_K(u) = \frac{p\phi(u)}{1 - q\phi(u)\, \phi_F(u)}$$

$$\Phi(u) = \frac{p\phi(u)\, \phi_F(u)}{1 - q\phi(u)\, \phi_F(u)} \; ;$$

if $T_F = \tau$, a constant, then

$$\Phi(u) = \frac{p\phi(u)\, e^{iu\tau}}{1 - q\phi(u)\, e^{iu\tau}} \quad .$$

Example: Let $X \sim ned(r)$; $f_{T_F}(t) = \frac{1}{\tau} e^{-t/\tau}$.

A3 $\qquad \Phi(u) = \dfrac{pr}{(r - iu)(1 - i\tau u) - qr}$

* $$h(t) = \frac{2pr\tau\, e^{-(1+r\tau)\frac{t}{2\tau}}}{\sqrt{4pr\tau - (1 + r\tau)^2}}\ \sin\left[\sqrt{4pr\tau - (1 + r\tau)^2}\ \frac{t}{2\tau}\right]$$

where $4pr\tau > (1 + r\tau)^2$

$$= \frac{2pr\tau e^{-(1+r\tau)\frac{t}{2\tau}}}{\sqrt{(1 + r\tau)^2 - 4pr\tau}}\ \sinh\sqrt{(1 + r\tau)^2 - 4pr\tau}\ \frac{t}{2\tau}$$

where $(1 + r\tau)^2 > 4pr\tau$.

C. <u>LIMITED AMMUNITION - RANDOM INDEPENDENT AMMUNITION RESUPPLY DELAY PROCEDURE (IFT AND TOF ALTERNATE)</u>

k_0 = initial ammunition supply (fixed)

k = replenishment supply (fixed and same each time)

Replenishments arrive randomly and independently of the firing process. Inter-arrival times, T_W, are ned(r_W). The subscript F refers to T_F (TOF). Let T_E = rv time during which ammunition is exhausted with cf $\phi_E(u)$ for $f_{T_E}(t)$

$$\phi_E(u) = \frac{C_1(u)^{k_0}}{r_W[1 - C_1(u)^k] - iu}\quad\text{where } C_1(u) \text{ is obtained from:}$$

$$C_1(u) = \phi_F\left\{r_W[1 - C_1(u)^k] - iu\right\}\ \phi\left\{r_W[1 - C_1(u)^k] - iu\right\}$$

$$\psi(u,y_F) = \frac{\phi_F(u)[\ 1 + iu\ \phi_E(u)]}{1 - \phi_F(u)\phi(u)}\ e^{iuy_F - \int_0^{y_F} \lambda_F(\xi)d\xi}$$

$$\psi(u,y) = \left\{ U(y) + \left[\frac{\phi_F(u)\ \phi(u) + iu\phi_E(u)}{1 - \phi_F(u)\ \phi(u)} \right] \right\} e^{iuy - \int_0^y \lambda(\xi)d\xi}$$

$$\Phi_K(u) = p \left\{ \frac{1 + iu\ q^{k_0}\left[\dfrac{C_2^{k_0}(u)}{r_W[1 - q^k\ C_2^k(u)] - iu} \right]}{1 - q\phi_F(u)\ \phi(u)} \right\} \phi(u)$$

$$\Phi(u) = \Phi_K(u)\ \phi_F(u) \quad \text{where} \quad C_2(u) \quad \text{comes from:}$$

$$C_2(u) = \phi_F\left\{ r_W[1 - q^k\ C_2^k(u)] - iu \right\} \phi\left\{ r_W[1 - q^k\ C_2^k(u)] - iu \right\}$$

1. <u>Special Case 1</u>:

No replenishment and ammunition is limited to k rounds
(n.b., can run out).

$$\Phi_K(u) = \frac{p\phi(u)}{1 - q\phi_F(u)\ \phi(u)}\ \left\{ 1 - [q\phi_F(u)\ \phi(u)]^k \right\} + q^k = \Phi_{K1}(u) + q^k$$

$$\Phi(u) = \Phi_{K1}(u)\ \phi(u) + q^k \ .$$

2. <u>Special Case 2</u>:

No replenishment. Ammunition limitation is a rv, i.e.

$$P[I = i] = \alpha_i ; \qquad \sum_{i=0}^{\infty}\ \alpha_i\ q^i \ . \quad \text{Can run out.}$$

FM - CRIFT

$$\Phi_K(u) = \frac{p\phi(u)}{1 - q\phi_F(u)\,\phi(u)} \left\{ 1 - \sum_{i=0}^{\infty} \alpha_i [q\phi_F(u)\,\phi(u)]^2 \right\} + \sum_{i=0}^{\infty} \alpha_i\, q^i$$

$$= \Phi_{K1}(u) + \sum_{i=0}^{\infty} \alpha_i\, q^i$$

$$\Phi(u) = \Phi_{K1}(u)\,\phi(u) + \sum_{i=0}^{\infty} \alpha_i\, q^i \ .$$

3. <u>Special Case 3</u>: Flight time is zero.

$$\Phi(u) = p\phi(u) \left\{ \frac{1 + iuq^{k_0}\left[\dfrac{C_3^{k_0}(u)}{r_W[1 - q^k C_3^k(u)] - iu} \right]}{1 - q\phi(u)} \right\} \quad \text{where } C_3(u) \text{ comes from:}$$

$$C_3(u) = \phi\{ r_W[1 - q^k C_3^k(u)] - iu \} \ .$$

<u>Example</u>: Let $X \sim$ ned(r) and let $(qC_3(u))^k \approx 0$, then

$$C_3(u) \simeq \frac{r}{r + r_W - iu}$$

J & H1 $$\Phi(u) = \frac{pr}{pr - iu} \left\{ 1 + \frac{iu}{r_W - iu} \left[\frac{qr}{r + r_W - iu} \right]^{k_0} \right\}$$

X.   <u>BURST FIRING</u>

Let

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T = rv time between bursts

T_G = rv time between rounds in a burst (burst intervals)

z = number of rounds in a burst (fixed)

$f(t)$ = pdf of T; cf of $f(t)$ = $\phi(u)$

$f_G(t)$ = pdf of T_G; cf of $f_G(t)$ = $\phi_G(u)$

$$\Psi(u,y) = \left[U(y) + \frac{q^z \phi(u) \phi_G^{z-1}(u)}{1 - q^z \phi(u) \phi_G^{z-1}(u)} \right] e^{iuy - \int_0^y \lambda(\xi)d\xi}$$

$$\Psi(u,y_G) = \frac{q\phi(u)[1 - (q\phi_G(u))^{z-1}]}{1 - q\phi(u)[1 - q^z \phi(u) \phi_G^{z-1}(u)]} \, e^{iuy_G - \int_0^{y_G} \lambda_G(\xi)d\xi} \qquad Bh(3)2$$

$$\Phi(u) = \frac{p\phi(u)[1 - q^z \phi_G^z(u)]}{[1 - q\phi_G(u)][1 - q^z \phi(u) \phi_G^{z-1}(u)]} \qquad \begin{array}{l} A5, \\ Bh3(2) \end{array}$$

If T_G = a, a constant, then

$$\Phi(u) = \frac{p\phi(u)[1 - q^z \exp(iauz)]}{[1 - q \exp(iau)][1 - q^z \phi(u) \exp(ia(z-1)u)]} \qquad A5$$

XI. MULTIPLE WEAPONS

A. FIRED IN VOLLEYS OF v ROUNDS EACH

p = P[Volley hits]; X = rv IFT, between volleys

d = P[round in volley kills | volley hits], same for all rounds in volley

$$\Phi(u) = \frac{[1 - (1 - d)^v] p\phi(u)}{1 - [q + (1 - d)^v p]\phi(u)}$$

Example: Let $X \sim \text{ned}(r)$,

$$\Phi(u) = \frac{pr[1 - (1 - d)^V]}{pr[1 - (1 - d)^V] - iu} \quad .$$

B. FIRED IN VOLLEYS OF v ROUNDS EACH WITH LIMITED AMMUNITION

$$P[I = i] = \alpha_i ; \qquad i = 1, 2, \ldots, \quad .$$

$$\Phi_1(u) = \sum_{i=1}^{\infty} \left[\frac{1 - (1 - d)^V}{(1 - d)^V} \right] \left[\frac{(1 - d)^V \, p\phi(u)}{1 - q\phi(u)} \right]^i$$

$$\cdot \left\{ \sum_{j=i}^{\infty} \alpha_i (1 - I_{q\phi(u)}(j - i + 1, i)) \right\}$$

** $$P[\bar{H}] = \sum_{i=0}^{j-1} \alpha_i + \sum_{i=j}^{\infty} \alpha_i \sum_{v=0}^{j-1} \binom{i}{v} p^v q^{i-v} [(1 - d)^V]^v \quad .$$

Example: Let $\alpha_i = (1 - \alpha)\alpha^i$ and $X \sim \text{ned}(r)$,

$$\Phi_1(u) = \frac{\alpha \, pr[1 - (1 - d)^V]}{r\{1 - \alpha[1 - p(1 - (1 - d)^V)]\}}$$

Kw & Bl $$P[\bar{H}] = \frac{1 - \alpha}{1 - \alpha[1 - p(1 - d)^V]} \quad .$$

C. MULTIPLE WEAPONS - USED SIMULTANEOUSLY

k weapons fired simultaneously, each weapon

$$X_i \sim \text{ned}(r_i) \quad \text{and} \quad P[H] = p_i, \quad i = 1,2,\ldots,k$$

$$P[t < T_M < t + dt, \text{ i-th weapon killed}] = p_i r_i e^{-p_i r_i t} dt$$

$$h(t) = \sum_{i=1}^{k} p_i r_i e^{-\sum_{i=1}^{k} p_i r_i t} \qquad \text{Bh1}$$

D. MULTIPLE WEAPONS - USED ALTERNATELY

Marksman fires 2 weapons alternately

Weapon 1: k_1 rounds fired each time with IFT $f_{X_1}(x_1)$

Weapon 2: k_2 rounds fired each time with IFT $f_{X_2}(x_2)$

Weapons have p_1, p_2 kill probabilities, respectively

Start with weapons unloaded

y_1 = time since weapon 1 fired last (no hits)

y_2 = time since weapon 2 fired last (no hits)   Bh2

$$\Psi(u,y_1) = \left\{ \begin{aligned} &(y_1)[1 - q_1\phi_1(u)]\{1 - [q_1\phi_1(u)]^{k_1} [q_2\phi_2(u)]^{k_2}\} + q_1\phi_1(u) \\ &-[q_1\phi_1(u)]^{k_1+1} [q_2\phi_2(u)]^{k_2} - [q_1\phi_1(u)]^{k_1} + [q_1\phi_1(u)]^{k_1}[q_2\phi_2(u)]^{k_2} \end{aligned} \right\} *$$

$$\frac{}{[1 - q_1\phi_1(u)]\{1 - [q_1\phi_1(u)]^{k_1} [q_2\phi_2(u)]^{k_2}\}}$$

$$\cdot \; e^{iuy_1 - \int_0^{y_1} \lambda(\xi)d\xi}$$

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$$\Psi(u,y_2) = \frac{[q_1\phi_1(u)]^{k_1}}{[1 - q_2\phi_2(u)]}$$

$$\cdot \left\{ \frac{1 - [q_2\phi_2(u)]^{k_2}}{1 - [q_1\phi_1(u)]^{k_1}[q_2\phi_2(u)]^{k_2}} \right\} e^{iuy_2 - \int_0^{y_2} \lambda_2(\xi)d\xi}$$

$$\Phi(u) = \frac{\left[\begin{array}{c} p_1\phi_1(u)\{1 - [q_1\phi_1(u)]^{k_1}\}[1 - q_2\phi_2(u)] + p_2\phi_2(u) \\ \cdot [q_1\phi_1(u)]^{k_1}\{1 - [q_2\phi_2(u)]^{k_2}\}[1 - q_1\phi_1(u)] \end{array} \right]}{[1 - q_1\phi_1(u)][1 - q_2\phi_2(u)]\{1 - [q_1\phi_1(u)]^{k_1}[q_2\phi_2(u)]^{k_2}\}} \cdot$$

Example: Let $k_1 = k_2 = 1$; $X_1, X_2 \sim r_1, r_2$, respectively.

Bh2
$$\Phi(u) = \frac{r_1 r_2(1 - q_1 q_2) - ip_1 r_1 u}{-u^2 - iu(r_1 + r_2) + r_1 r_2(1 - q_1 q_2)} \cdot$$

E. **MULTIPLE WEAPONS USED CONSECUTIVELY - EACH USED UNTIL FAILURE**

1. **Marksman has k Rounds Initially (Ammunition Limitation) and m Weapons**

   Let $X \sim ned(r)$ and $T_L \sim rv$ time to failure; same for each weapon when in use $\sim ned(r_L)$.

$$\Phi_0(u) = \left(\frac{qr}{r - iu} \right)^k I_{\left[\frac{r-iu}{r+r_L-iu} \right]}(k,m) + \left(\frac{r_L}{pr + r_L - iu} \right)^m I_{\left[\frac{pr+r_L-iu}{r + r_L - iu} \right]}(m,k)$$

$$\Phi_1(u) = \frac{pr}{pr - iu}$$

$$\cdot \left[1 - \left(\frac{qr}{r - iu} \right)^k I_{\left[\frac{r-iu}{r+r_L-iu} \right]}(k,m) - \left(\frac{r_L}{pr + r_L - iu} \right)^m \right.$$

$$\left. \cdot I_{\left[\frac{pr+r_L-iu}{r+r_L-iu} \right]}(m,k) \right] \quad .$$

2. Unlimited Ammunition, Random Initial Supply of Weapons

$$X \sim \text{ned}(r); \quad T_L \sim \text{ned}(r_L)$$

$$P[M = m] = \alpha_m \; ; \qquad \sum_{m=0}^{\infty} \alpha_m = 1$$

$$\Phi_0(u) = \sum_{m=0}^{\infty} \alpha_m \left(\frac{r_L}{pr + r_L - iu} \right)^m$$

$$\Phi_1(u) = \frac{pr}{pr - iu} \left[1 - \sum_{m=0}^{\infty} \alpha_m \left(\frac{r_L}{pr + r_L - iu} \right)^m \right] \quad .$$

Example:

$$\alpha_m = (1 - \alpha)\alpha^m$$

$$\Phi_0(u) = \left(1 - \alpha\right)\left[\frac{pr + r_L - iu}{pr + (1 - \alpha)r_L - iu}\right]$$

$$\Phi_1(u) = \frac{pr\alpha}{pr + r_L(1 - \alpha) - iu} \quad .$$

3. **Unlimited Ammunition, Fixed Initial Supply of Weapons**

$X \sim ned(r); \qquad T_L \sim ned(r_L); \quad m \quad$ number of weapons

$$\Phi_0(u) = \left(\frac{r_L}{pr + r_L - iu}\right)^m$$

Bh6 $\qquad \Phi_1(u) = \frac{pr}{pr - iu}\left[1 - \left(\frac{r_L}{pr + r_L - iu}\right)^m\right] \quad .$

XII. **MARKOV-DEPENDENT FIRE**

See FM-FIFT for notation.

A. **POSITIVELY CORRELATED FIRE**

Three State Firer

$$P[H_i \mid H_{i-1}] = p_0; \quad P[H_i \mid \bar{H}_{i-1}] = p_1; \qquad p_0 > p_1$$

$$P[H_i] = p = \frac{p_1}{1 - p_0 + p_1} \; ; \quad \rho = \text{Corr}[H_i, H_{i+1}] = p_0 - p_1$$

$$P[K \mid H] = p_k$$

$$
\underset{\sim}{S} \;=\; \begin{matrix} \\ \bar{H} \\ \\ H\bar{K} \\ \\ K \end{matrix}
\overset{\begin{matrix} \bar{H} & \quad H\bar{K} & \quad K \end{matrix}}{
\left(
\begin{array}{cc|c}
1 - p_1 & p_1(1 - p_k) & p_1 p_k \\[6pt]
1 - p_0 & p_0(1 - p_k) & p_0 p_k \\
\hline
0 & 0 & 1
\end{array}
\right)}
\;=\;
\left(
\begin{array}{c|c}
\underset{\sim}{P} & \underset{\sim}{k} \\
\hline
0 & 1
\end{array}
\right)
$$

Let

$$
\underset{\sim}{p}^{T} \;=\; (1 - p, \, p, \mid 0) \;=\; (\, \underset{\sim}{q}^{T} \mid 0 \,)
$$

$$
p_N(n) \;=\; \underset{\sim}{p}^{T} \, \underset{\sim}{S}^{n} \, \underset{\sim}{n} \;-\; \underset{\sim}{p}^{T} \, \underset{\sim}{S}^{n-1} \, \underset{\sim}{n}
$$

$$
E[N] \;=\; \underset{\sim}{q}^{T}(\underset{\sim}{I} - \underset{\sim}{P})^{-1} \, \underset{\sim}{e}_n
$$

$$
V[N] \;=\; \underset{\sim}{q}^{T}(2(\underset{\sim}{I} - \underset{\sim}{P})^{-1} - \underset{\sim}{I})(\underset{\sim}{I} - \underset{\sim}{P})^{-1} \, \underset{\sim}{n} \;-\; E^{2}[N]
$$

$$
E[T_M] \;=\; E[N] \cdot E[T]
$$

$$
V[T_M] \;=\; E[N] \cdot V[T] + V[N] \, E^{2}[T]
$$

$$
\Phi(u) \;=\; \frac{p_1 \, p_k}{1 - p_0 + p_k} \left(\frac{\phi(u)[1 - (p_0 - p_1)\phi(u)]}{1 - \phi(u)[1 - p_1 + p_0(1 - p_k)(1 - \phi(u))] + p_1(1 - p_k)\phi(u)} \right)
$$

Example: Let $X \sim \text{ned}(1)$, then

$$
\Phi(u) \;=\; \cdot\cdot\; \frac{C_1 \, C_2}{C_3} \; \frac{iu - C_3}{(iu - C_1)(iu - C_2)}
$$

where

$$C_1 = \frac{1}{2}\left[1 + p_1 - p_0(1 - p_k) + \sqrt{1 + p_1 - p_0(1 - p_k)^2 - 4p_1 p_k}\right]$$

$$C_2 = \frac{1}{2}\left[1 + p_1 - p_0(1 - p_k) - \sqrt{1 + p_1 - p_0(1 - p_k)^2 - 4p_1 p_k}\right]$$

Fil $\qquad C_3 = 1 - p_0 + p_1$.

B. <u>IFT'S ned</u>

pdf IFT when in state $E_i = r_i e^{-r_i t}$, $\quad t \geq 0$

$\qquad\qquad\qquad\qquad\qquad = \quad 0 \quad$, elsewhere

$$\underset{\sim}{D}(r_i) = \begin{pmatrix} r_0 & 0 & \cdots & \cdots & 0 \\ 0 & r_1 & 0 & \cdots & 0 \\ 0 & 0 & r_2 & \cdots & 0 \\ \vdots & & & r_i & \vdots \\ 0 & \cdots & & \cdots & r_m \end{pmatrix}$$

$\underset{\sim}{A} = \underset{\sim}{D}(r_i)(\underset{\sim}{S} - \underset{\sim}{I})$; $\quad H(t) = \underset{\sim}{m}^T \underset{\sim}{e}^{At} \underset{\sim}{n}$

$h(t) = \underset{\sim}{m}^T \underset{\sim}{e}^{At} \underset{\sim}{A} \underset{\sim}{n}$

$\underset{\sim}{\lambda}^T = (\lambda_1,\ldots,\lambda_i,\ldots,\lambda_m)$ where $\lambda_i < 0$, $i = 1,2,\ldots,m$, $\lambda_0 = 0$; we note that the λ_i's are the characteristic values of $\underset{\sim}{A}$ and <u>if</u> $\underset{\sim}{A}$ has $m+1$ linearly independent characteristic vectors, then let

$\underset{\sim}{X}$ be the matrix of characteristic vectors of $\underset{\sim}{A}$

$\underset{\sim 0}{x}^T$ be the zeroeth row of $\underset{\sim}{X}$

$\underset{\sim m}{x}'$ be the m-th column of $\underset{\sim}{X}^{-1}$

$$H(t) = 1 + \underset{\sim 0}{x}^T \underset{\sim}{D}(e^{\lambda_1 t}) \underset{\sim m}{x}'$$

$$h(t) = \underset{\sim 0}{x}^T \underset{\sim}{D}(\lambda_1 e^{\lambda_1 t}) \underset{\sim m}{x}' .$$ Ba3

C. MULTIPLE WEAPONS: TWO WEAPONS FIRED IN RANDOM MARKOV-DEPENDENT ORDER

Weapon 1: $\begin{cases} \text{IFT} - X_1 \\ \\ \text{Hit Probability} = p_1; \quad q_1 = 1 - p_1 \end{cases}$

Weapon 2: $\begin{cases} \text{IFT} - X_2 \\ \\ \text{Hit Probability} = p_2 \end{cases}$

Firing order determined by transition matrix

$$\underset{\sim}{F} = \begin{array}{c} \\ 1 \\ 2 \end{array} \overset{\text{Weapons} \quad 1 \quad\quad 2}{\left(\begin{array}{cc} p_{11} & p_{12} \\ p_{21} & p_{22} \end{array} \right)}, \text{ firing starts with Weapon 1}$$

Let

$$C = 1 - q_1 p_{11} \phi_1(u) - q_2 p_{22} \phi_2(u) - q_1 q_2 (p_{11}p_{22} - p_{12}p_{21})\phi_1(u) \phi_2(u)$$

$$\Theta_0(u) = \frac{q_1 p_{11} \phi_1(u) + q_1 q_2 (p_{11}p_{22} - p_{12}p_{21})\phi_1(u) \phi_2(u)}{C}$$

C67

$$\psi(u,y_1) = [U(y_1) + \Theta_0(u)]e^{iuy_1 - \int_0^{y_1}\lambda_1(\xi)d\xi}$$

$$\psi(u,y_2) = \frac{p_{12}q_1\phi_1(u)}{C} e^{iuy_2 - \int_0^{y_2}\lambda_2(\xi)d\xi}$$

$$\Phi(u) = \frac{\phi_1(u)}{C}\left[P_1 + (p_1q_2p_{22} - q_1p_2p_{12})\phi_2(u)\right] .$$

Example: Let $X_1 \sim \text{ned}(r_1)$ and $X_2 \sim \text{ned}(r_2)$.

$$\Phi(u) = \frac{C_1C_2 - ip_1r_1u}{(C_1 - iu)(C_2 - iu)}$$

where

$$C_1 + C_2 = r_1(1 - q_1p_{11}) + r_2(1 - q_2p_{22})$$

Bh8
$$C_1 C_2 = r_1r_2[1 - q_1p_{11} - q_2p_{22} + q_1q_{12}(p_{11}p_{22} - p_{12}p_{21})] .$$

XIII. MISCELLANEOUS RESULTS

THEOREM: Let $P[H] = p$, $P[\bar{H}] = 1 - p$, $P[K|H] = p_k$ and p, p_k constants; however, conditional hit probabilities vary depending on prior history. Define the positive correlation as:

$P[H_i \mid$ hits on specified previous rounds and miss on all others]

$\geq P[H_i \mid$ miss on at least one of the specified previous rounds and miss on all others] .

Then, firer always does worse with positive correlation than with independent firing.

n.b. (a) any hit may be a kill, i.e., overkilling is allowed

(b) this is not the same as the usual definition of correlation
(for which this theorem does not apply). It is stronger
than ordinary correlation.

This theorem does not apply, in general, to FM or FD but may be
specialized to apply by making the first kill an absorbing state. Fil

FD-FHFT

FUNDAMENTAL DUEL - FIXED INTERFIRING TIMES

(FD - FIFT)

I. FD - FIFT

$$X_A = a_1; \quad X_B = b_1; \quad \frac{a_1}{b_1} = \frac{a}{b} \text{ where } a, b \text{ are relatively prime integers}$$

$$\text{and } a_1/b_1 \text{ is rational}$$

$$m = \left[\frac{a}{b}\right]; \quad [x_j] = \left[(j + 1)\frac{R}{b}\right]$$

$$R = a - b\left[\frac{a}{b}\right]; \quad 0 < R < b, \quad a = mb + R$$

$$P(A) = \frac{p_A q_B^m}{1 - q_A^b q_B^a} \sum_{j=0}^{b-1} q_A^j \, q_B^{jm + [x_j]}$$

$$= \frac{p_A}{1 - q_A^b q_B^a} \sum_{j=0}^{b-1} q_A^j q_B^{\left[(j+1)\frac{a}{b}\right]}$$

$$P(AB) = \frac{p_A p_B \, q_A^{b-1} q_B^{a-1}}{1 - q_A^b q_B^a}$$

A&W1

$$P(N_A = n \mid A) = \frac{p_A \, q_A^{n-1}}{P(A)} \, q_B^{\left[n\frac{a}{b}\right]}$$

$$E(N_A \mid A) = \frac{b \, q_A^b q_B^a}{1 - q_A^b q_B^a} + \frac{\sum\limits_{j=0}^{b-1} (j + 1)(q_A q_B^m)^j \, q_B^{[x_j]}}{\sum\limits_{j=0}^{b-1} (q_A q_B^m)^j \, q_B^{[x_j]}}$$

$$E(N_A^2 \mid A) = \frac{p_A \, q_B^m}{P(A)(1 - q_A^b q_B^a)} \left\{ \frac{b^2 \, q_A^b q_B^a (1 + q_A^b q_B^a)}{(1 - q_A^b q_B^a)^2} \sum_{j=0}^{b-1} (q_A q_B^m)^j \, q_B^{[x_j]} + \right.$$

$$+ \frac{2b\, q_A^b q_B^a}{1 - q_A^b q_B^a} \sum_{j=0}^{b-1} (j+1)(q_A q_B^m)^j\, q_B^{[x_j]}$$

$$+ \sum_{j=0}^{b-1} (j+1)^2 (q_A q_B^m)^j\, q_B^{[x_j]} \Bigg\}$$

$$P(N_A \geq n_0 \mid A) = \frac{p_A (q_A^b q_B^a)^{[n_0/b]}}{P(A)(1 - q_A^b q_B^a)}$$

$$\cdot \left[q_A^b q_B^a \sum_{j=0}^{\lambda-1} (q_A q_B^m)^j\, q_B^{[x_j]} + \sum_{j=\lambda}^{b-1} (q_A q_B^m)^j\, q_B^{[x_j]} \right]$$

where $\lambda = n_0 - [n_0/b]b$, i.e., the remainder when n_0 is divided by

b, $0 \leq \lambda < b$

$$P(A)\, P(T_D = na_1 \mid A) = p_A\, q_A^{n-1}\, q_B^{[na/b]} , \qquad n = 1,2,\ldots$$

$$P(AB)\, P(T_D = nba_1 \mid AB) = \frac{p_A p_B}{q_A q_B} (q_A^b q_B^a)^n , \qquad n = 1,2,\ldots$$

$$g(t) = P[A]\, P[T_D = na_1 \mid A]\, \delta(t - na_1) + P[B]\, P[T_D = nb_1 \mid B]\, \delta(t - nb_1)$$

$$+ P[AB]\, P[T_D = nba_1 \mid AB]\, \delta(t - nba_1)$$

Example 1: Let $a_1 = cb_1$ with $a = c$, $b = 1$ and c a positive integer:

C72

$$P(A) = \frac{p_A \, q_B^c}{1 - q_A q_B^c} \qquad \text{and} \qquad P(AB) = \frac{p_A p_B \, q_B^{c-1}}{1 - q_A q_B^c} \ .$$

Example 2: Let $b_1 = ca_1$ with $a = 1$, $b = c$ and c a positive integer;

$$P(A) = 1 - \frac{q_A^{c-1} \, p_B}{1 - q_A^c q_B} \qquad \text{and} \qquad P(AB) = \frac{p_A p_B \, q_A^{c-1}}{1 - q_A^c q_B} \ . \qquad \text{A \& W1}$$

Example 3:

On the following page are three graphs, as follows:

(a) From Example 1 with $c = 2$. This is the same as $r_A = \frac{1}{2} r_B$. The right end points of each contour are at $p_B = 1 - \sqrt{P(A)}$.

(b) From Example 1 with $c = a = b = 1$. This is the same as $r_A = r_B$. The right end points of each contour are at $p_B = 1 - P(A)$.

(c) From Example 2 with $c = 2$. This is the same as $r_A = 2r_B$. The end points of all contours are at $p_A = P(A)$.

(b)

(a)

(c)

A & W1   The Fundamental Duel with Discrete Firing Times
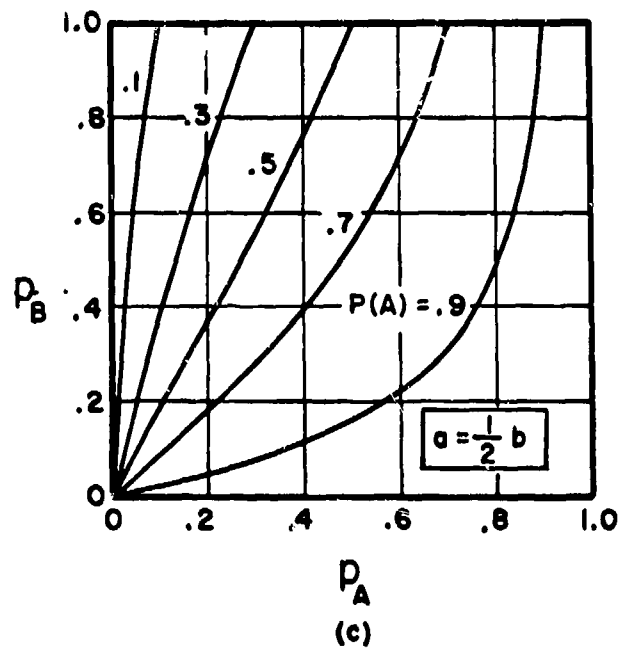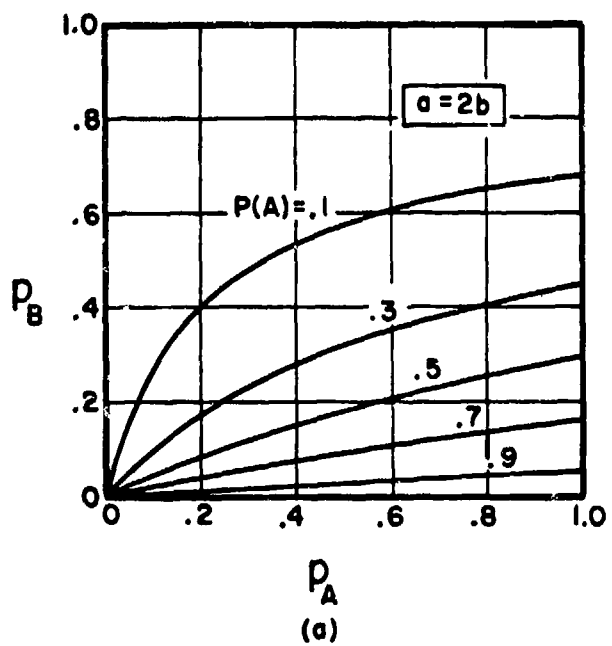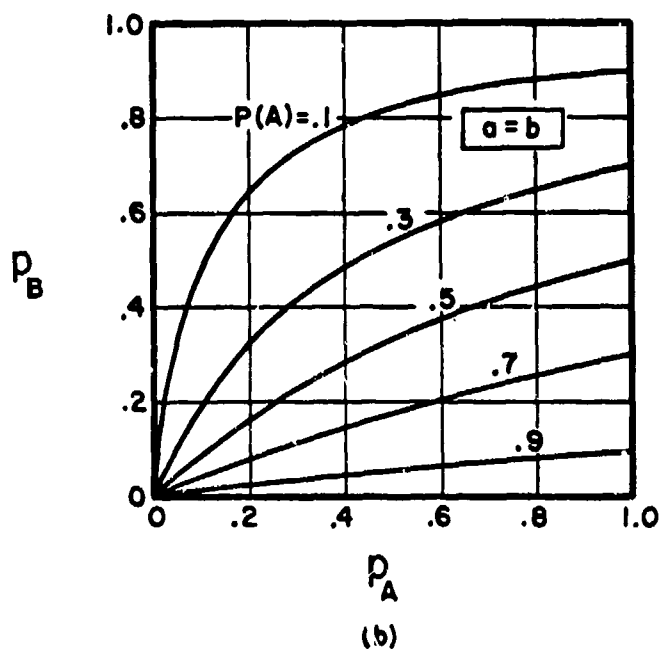
II. VARIATIONS OF INITIAL CONDITIONS - INITIAL SURPRISE

A has time t_s in which to fire before B starts

$\lambda =$ cycle time $= a_1 b = a b_1$

$y =$ number of rounds fired by A before B's first round

$$= \left[\frac{t_s}{a_1} + 1 \right] \quad \text{where} \quad [x] = \text{largest integer} \leq x$$

In λ time units A will fire b rounds and B will fire a rounds

Let $c_1 = q_A^y$, $c_2 = 1 - q_A^y$

In a cycle there may be either 0 or 1 simultaneous firings.

Let $M = a + b$, and define

$$
\underset{\sim}{T} = \begin{cases}
\underset{\sim}{T}_{y+1} \; \underset{\sim}{T}_{y+2} \; \cdots \; \underset{\sim}{T}_{y+i} \; \cdots \; \underset{\sim}{T}_{y+M} \quad , & \text{if no simultaneous firings} \\[3ex]
\underset{\sim}{T}_{y+1} \; \cdots\cdots\cdots\cdots\cdots \; \underset{\sim}{T}_{y+M-1} \; , & \text{if one simultaneous firing}
\end{cases}
$$

$$
= \begin{pmatrix}
t_1 & t_2 & t_3 & t_4 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

where $\underset{\sim}{T}_{y+i} = \underset{\sim}{T}_A$ or $= \underset{\sim}{T}_B$ or $= \underset{\sim}{T}_{AB}$, depending on the order of firing in the first cycle, $i = 1, 2, \ldots, M$ (or $M-1$).

$$T_A = \begin{pmatrix} q_A & p_A & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad T_B = \begin{pmatrix} q_B & 0 & p_B & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_{AB} = \begin{pmatrix} q_A q_B & p_A q_B & p_B q_A & p_A p_B \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad .$$

The firing order in the first cycle is determined by:

$$t_{Ai} = \text{time } A \text{ fires } i\text{-th round} = (i - 1)a_1$$

$$t_{Bj} = \text{time } B \text{ fires } j\text{-th round} = t_s + (j - 1)b_1$$

(n.b., A fires b rounds in one cycle; B fires a rounds in one cycle; and B fires his first round at time $= t_s$).

After j Cycles

$$P^j[\text{no hits}] = c_1 t_1^j$$

$$P^j[A] = c_2 + \frac{c_1 t_2}{1 - t_1} (1 - t_1^j)$$

$$P^j[B] = \frac{c_1 t_3}{1 - t_1} (1 - t_1^j)$$

$$P^j[AB] = \frac{c_1 t_4}{1 - t_1} (1 - t_1^j) \quad .$$

$$P(A) = 1 - q_A^y + \frac{q_A^y \, t_2}{1 - t_1}$$

$$P(B) = \frac{q_A^y \, t_3}{1 - t_1}$$

$$P(AB) = \frac{q_A^y \, t_4}{1 - t_1} \quad . \qquad\qquad\qquad\qquad Gr1$$

Example: Let $a = b$ with $t_s < a$,

$$P(A) = \frac{p_A}{1 - q_A q_B} \qquad \text{and} \qquad P(B) = \frac{q_A p_B}{1 - q_A p_B}$$

$$P[(k-1)a + t_s < T_b < ka + t_s, \ A \text{ is alive}]$$

$$= (q_A q_B)^k + p_A \sum_{j=0}^{k-1} (q_A q_B)^j$$

$$P[(k-1)a < T_b < ka, \ B \text{ is alive}] \qquad\qquad Sc1$$

$$= q_A \left[(q_A q_B)^{k-1} + p_B \sum_{j=0}^{k-2} q_A q_B^j \right] \quad .$$

III. MULTIPLE HITS TO A KILL

Alternate firing, i.e., $a_1 = b_1$; $t_s < a_1$ where:

R_i = fixed number of hits required for i to kill j (either may start first), $i, j = A, B$

$P[i \mid j]$ = P[i wins given j starts first], $i, j = A, B$

Let $C = \min (R_i, R_j - 1)$ where i is the contestant to start first

$$(A \quad \text{or} \quad B)$$

$$P[B \mid A] = \sum_{i=0}^{C} \sum_{j=0}^{R_B - i - 1} \binom{R_A}{i} \binom{R_A + j - 1}{j} p_A^{i+j} p_B^{R_A} q_A^{R_A - 1} q_B^{j} (1 - q_A q_B)^{-R_A - j}$$

$$P[A \mid A] = 1 - P[B \mid A]$$

To obtain the other probabilities, interchange A and B.

Examples:

R_A	R_B	$p_A = .3$ $p_B = .5$		$p_A = p_B = .5$		$p_A = .5$ $p_B = .7$	
		$P[B \mid A]$	$P[B \mid B]$	$P[B \mid A]$	$P[B \mid B]$	$P[B \mid A]$	$P[B \mid B]$
1	1	.5385	.7692	.3333	.6667	.4118	.8235
5	3	.4257	.5010	.1139	.1728	.2576	.3579
5	5	.8201	.8630	.4512	.5488	.7414	.8381
7	5	.5955	.6541	.1674	.2266	.4159	.5278
7	7	.8695	.8981	.4599	.5401	.7981	.8669
10	10	.9160	.9330	.4671	.5329	.8545	.9002

Z1

IV. LIMITED AMMUNITION

A draw occurs if both run out of ammunition, or if A and B kill simultaneously.

A. <u>AMMUNITION SUPPLY A RV</u>

Let $X_A = a_1$ and $X_B = b_1$;

$$P(I = i) = \alpha_i , \quad P(\tau = \infty) = \alpha_\infty , \quad i = 1,2,\ldots, \text{ and } \alpha_\infty + \sum_{i=0}^{\infty} \alpha_i = 1$$

$$P(J = j) = \beta_j , \quad P(J = \infty) = \beta_\infty , \quad j = 1,2,\ldots, \text{ and } \beta_\infty + \sum_{j=0}^{\infty} \beta_j = 1$$

$$P(A) = \sum_{n=1}^{\infty} \left[p_A q_A^{n-1} \left(\alpha_\infty + \sum_{i=n}^{\infty} \alpha_i \right) \right.$$

$$\left. \cdot \left[\sum_{j=0}^{[na/b]} \beta_j q_B^j + \left(\beta_\infty + \sum_{j=[na/b]+1}^{\infty} \beta_j \right) q_B^{[na/b]} \right] \right]$$

$$P(AB) = \sum_{i=0}^{\infty} \alpha_i q_A^i \sum_{j=0}^{\infty} \beta_j q_B^j + p_A p_B \sum_{\nu=1}^{\infty} q_A^{\nu b-1} q_B^{\nu a-1}$$

$$\cdot \left(\alpha_\infty + \sum_{i=\nu b}^{\infty} \alpha_i \right) \left(\beta_\infty + \sum_{j=\nu a}^{\infty} \beta_j \right)$$

$$P(N_A = n \mid A) = \frac{p_A q_A^{n-1}}{P(A)} \left(\alpha_\infty + \sum_{i=n}^{\infty} \alpha_i \right)$$

$$\sum_{j=0}^{[na/b]} \beta_j q_B^j + \left(\beta_\infty + \sum_{j=[na/b]+1}^{\infty} \beta_j \right) q_B^{[na/b]} .$$

Occasionally, in what follows, there are sums which may have upper limits which are less than the lower limits, under certain conditions. In all such cases the sum is to be considered zero.

$$P(N_A = n \mid B) \, P(B) = \alpha_0 \Big(1 - \sum_{j=0}^{\infty} q_B^j \beta_j \Big) + (1 - \alpha_0) \, p_B \sum_{k=1}^{a-1} q_B^{k-1} \Big(\beta_\infty + \sum_{j=k}^{\infty} \beta_j \Big), \quad b = 1$$

$$= \alpha_0 \Big(1 - \sum_{j=0}^{\infty} q_B^j \beta_j \Big) + (1 - \alpha_0) p_B \sum_{k=1}^{[a/b]} q_B^{k-1} \Big(\beta_\infty + \sum_{j=k}^{\infty} \beta_j \Big), \quad b \geq 2$$

where $n = 0$.

$$P(N_A = n \mid B) \, P(B)$$

$$= q_A^n \Bigg\{ \alpha_n \, q_B^{[na/b]} \, p_B \sum_{k=1}^{\infty} q_B^{k-1} \Big(\beta_\infty + \sum_{j=k}^{\infty} \beta_{[na/b]+j} \Big) + \Big(\alpha_\infty \sum_{i=n+1}^{\infty} \alpha_i \Big) q_B^{[na/b]}$$

$$\cdot \, p_B \sum_{k=1}^{[(n+1)a/b]-[na/b]} q_B^{k-1} \Big(\beta_\infty + \sum_{j=k}^{\infty} \beta_{[na/b]+j} \Big) \Bigg\}$$

where $n \neq 0, b, 2b, \ldots,$ and $n+1 \neq 0, b, 2b, \ldots$,

$$P[N_A = n \mid B) \, P(B)$$

$$= q_A^n \Bigg\{ \alpha_n \, q_B^{[na/b]} \, p_B \sum_{k=1}^{\infty} q_B^{k-1} \Big(\beta_\infty + \sum_{j=k}^{\infty} \beta_{[na/b]+j} \Big) + \Big(\alpha_\infty + \sum_{i=n+1}^{\infty} \alpha_i \Big) q_B^{[na/b]} \, .$$

$$\cdot \; p_B \sum_{k=1}^{[a/b]-1} q_B^{k-1} \left(\beta_\infty + \sum_{j=k}^{\infty} \beta_{[na/b]+j} \right\}$$

where $n \neq 0, b, 2b, \ldots,$ and $n + 1 = b, 2b, \ldots$.

$$P(N_A = n \mid B) \; P(B)$$

$$= q_A^n \left\{ \alpha_n \, q_B^{(na/b)-1} \, p_B \sum_{k=1}^{\infty} \left(\beta_\infty + \sum_{j=k}^{\infty} \beta_{(na/b)+j-1} \right) \right.$$

$$+ \left(\alpha_\infty + \sum_{i=n+1}^{\infty} \alpha_i \right) q_B^{(na/b)-1} \, p_B$$

$$\cdot \sum_{k=1}^{(a/b)+1} q_B^{k-1} \left(\beta_\infty + \sum_{j=k}^{\infty} \beta_{(na/b)-1+j} \right) \right\}$$

where $n = b, 2b, \ldots$.

$$P(N_A = n \mid B) \; P(B)$$

$$= q_A^n \left\{ \alpha_n \, q_B^{na-1} \, p_B \sum_{k=1}^{\infty} q_B^{k-1} \left(\beta_\infty + \sum_{j=k}^{\infty} \beta_{na+j-1} \right) \right.$$

$$+ \left(\alpha_\infty + \sum_{i=n+1}^{\infty} \alpha_i \right) q_B^{na-1} \, p_B \sum_{k=1}^{a-1} q_B^{k-1} \; .$$

$$\cdot \left(\beta_\infty + \sum_{j=k}^\infty \beta_{na+j} \right) \Bigg\}, \quad n \geq 1 \quad \text{and} \quad b = 1$$

$$P(N_A = n \mid AB) = \frac{\alpha_n \, q_A^n}{P(AB)} \sum_{j=0}^\infty \beta_j \, q_B^j, \quad \text{where} \quad n \neq b, 2b, \ldots \; .$$

$$P(N_A = n \mid AB)$$

$$= \frac{1}{P(AB)} \Bigg\{ \alpha_n \, q_A^n \sum_{j=0}^\infty \beta_j \, q_B^j + p_A p_B \, q_A^{n-1} \, q_B^{(na/b)-1} \left(\alpha_\infty + \sum_{i=n}^\infty \alpha_i \right)$$

$$\cdot \left(\beta_\infty + \sum_{j=(na/b)}^\infty \beta_j \right) \Bigg\}$$

where $n = b, 2b, 3b, \ldots \; .$

$$P(N_A = n) = P(N_A = n \mid A) \, P(A) + P(N_A = n \mid B) \, P(B) + P(N_A = n \mid AB) \, P(AB) \; .$$

B. <u>FIXED AMMUNITION SUPPLY</u>

Let $\alpha_k = 1, \quad \alpha_\infty = \alpha_i = 0, \quad i \neq k \;$,

$\beta_\ell = 1, \quad \beta_\infty = \beta_j = 0, \quad j \neq \ell \; .$

$P(A) = q_B^\ell (1 - q_A^k) \;$,

$$n_1 = 0$$

$$P(A) = p_A \sum_{n=1}^{n_1} q_A^{n-1} q_B^{[na/b]} + q_B^{\ell}(q_A^{n_1} - q_A^k), \qquad 1 \le n_1 \le k$$

$$= p_A \sum_{n=1}^{k} q_A^{n-1} q_B^{[na/b]}, \qquad k \le n_1 \le k\ell$$

$$= 1 - q_A^k, \qquad n_1 > k\ell$$

where $n_1 = [\ell b/a]$

$$P(AB) = q_A^k q_B^{\ell} + p_A p_B \sum_{j=1}^{\min(\circ, \eta)} q_A^{jb-1} q_B^{ja-1}, \qquad k \ge b \ \text{and} \ \ell \ge a$$

$$= q_A^k q_B^{\ell}, \qquad k < b \ \text{or} \ \ell < a$$

where $\circ = [k/b]$ and $\eta = [\ell/a]$.

$\Delta P(A) = P(N_A \ge i+1, A) - P(N_A \ge j+1, A)$, using $\alpha_\infty = 1$,

where $\Delta P(A)$ is the increase in A's kill probability if A's

initial supply is increased from i to j.

$$P(A) \, P(N_A = n \mid A) = p_A \, q_A^{n-1} \, q_B^{\ell}$$

$$= p_A \, q_A^{n-1} \, q_B^{[na/b]}, \quad n \le n_1 \left.\vphantom{\begin{array}{c}a\\a\end{array}}\right\}$$

$$= p_A \, q_A^{n-1} \, q_B^{\ell}, \quad n > n_1 \left.\vphantom{\begin{array}{c}a\\a\end{array}}\right\} \ 1 \le n_1 \le k$$

$$= p_A \, q_A^{n-1} \, q_B^{[na/b]}, \qquad k \le n_1 \le k\ell$$

$$= p_A \, q_A^{n-1}, \qquad n_1 > k\ell$$

$$E(N_A | A) = \sum_{n=1}^{k} nP(N_A = n | A) = q_B^{\ell}\{kq_A^{k+1} - (k+1)q_A^k + 1\} , \qquad n_1 = 0$$

$$= p_A \sum_{n=1}^{n_1} nq_A^{n-1} q_B^{[na/b]} + q_B(q_A^{n_1} - q_A^k) \qquad , \qquad 1 \leq n_1 \leq k$$

$$= p_A \sum_{n=1}^{k} nq_A^{n-1} q_B^{[na/b]} \qquad , \qquad k \leq n_1 \leq k\ell$$

$$= kq_A^{k+1} - (k+1)q_A^k + 1 \qquad . \qquad n_1 \geq k\ell$$

$$P(N_A > n_0 | A) = \sum_{n=n_0}^{k} P(N_A = n | A) = q_B^{\ell}(q_A^{n_0-1} - q_A^k) , \qquad n_1 = 0$$

$$= p_A q_A^{n-1} \sum_{n=n_0}^{n_1} q_A^{n-1} q_B^{[na/b]} + q_B^{\ell}(q_A^{n_1} - q_A^k), \; n_0 \leq n_1$$

$$\left.\begin{array}{l} \\ \\ \end{array}\right\} 1 \leq n_1 \leq \ell$$

$$= q_B^{\ell}(q_A^{n_0-1} - q_A^k) \qquad , \; n_0 > n_1$$

$$= p_A \sum_{n=n_0}^{k} q_A^{n-1} q_B^{[na/b]} \qquad , \qquad k \leq n_1 \leq k\ell$$

$$= q_A^{n_0-1} - q_A^k \qquad . \qquad n_1 > k\ell$$

$$\Delta P(A) = P(N_A \geq i+1, A) - P(N_A \geq j+1, A), \quad \text{using} \quad \alpha_{\infty} = 1,$$

where $\Delta P(A)$ is the increase in A's kill probability if A's initial supply is increased from i to j.

Example: Let $m = [a/b]$; $[x_j] = [(j+1)\frac{R}{b}]$ where $R = a - bm$;

also let $\alpha_i = (1-\alpha)\alpha^i$, $i = 0,1,2,\ldots,$; $\alpha_\infty = 0$; $\beta_j = (1-\beta)\beta^j$,

$j = 0,1,2,\ldots,$ and $\beta_\infty = 0$.

$$P(A) = \frac{\alpha p_A}{1-\beta q_B} \left\{ \frac{1-\beta}{1-\alpha q_A} + \frac{\beta p_B (\beta q_B)^m}{1-(\alpha q_A)^b (\beta q_B)^a} \sum_{j=0}^{b-1} \left(\alpha q_A (\beta q_B)^m \right)^j (\beta q_B)^{[x_j]} \right\}$$

$$P(AB) = \left(\frac{1-\alpha}{1-\alpha q_A} \right) \left(\frac{1-\beta}{1-\beta q_B} \right) + \frac{p_A p_B (\alpha q_A)^b (\beta q_B)^a}{q_A q_B [1 - (\alpha q_A)^b (\beta q_B)^a]}$$

$$P(N_A = n | A) = \frac{\alpha p_A (\alpha q_A)^{n-1}}{P(A)(1 - \beta q_B)} \left\{ (1 - \beta) + \beta p_B (\beta q_B)^{[na/b]} \right\}$$

$$E(N_A | A) = \frac{\alpha p_A}{P(A)(1-\beta q_B)} \left\{ \frac{1-\beta}{(1-\alpha q_A)^2} + \frac{\beta p_B (\beta q_B)^m}{1 - (\alpha q_A)^n (\beta q_B)^a} \right.$$

$$\cdot \left[\frac{b(\alpha q_A)^b (\beta q_B)^a}{1 - (\alpha q_A)^b (\beta q_B)^a} \sum_{j=0}^{b-1} (\alpha q_A)^j (\beta q_B)^{mj+[x_j]} \right.$$

$$\left. \left. + \sum_{j=0}^{b-1} (j+1)(\alpha q_A)^j (\beta q_B)^{mj+[x_j]} \right] \right\} .$$

A & G1

V. TIME-LIMITATION

A draw occurs if time runs out or both kill simultaneously.

A. <u>TIME LIMIT A RV</u>

Let T_L = time limit rv where $f_{T_L}(t)$ is the pdf of T_L and let $X_A = a_1$, $X_B = b_1$.

$$P(A) = p_A \sum_{n=1}^{\infty} q_A^{n-1} q_B^{[na/b]} \int_{na_1}^{\infty} f_{T_L}(t)dt$$

$$P(AB_1) = \frac{p_A p_B}{q_A q_B} \sum_{n=1}^{\infty} (q_A^b q_B^a)^n \int_{nba_1}^{\infty} f_{T_L}(t)dt \quad \text{where} \quad AB_1 = \text{Event of}$$

simultaneous kills

$$P(AB_2) = \sum_{n=0}^{\infty} \left(q_B^n q_A^{[nb/a]} \int_{nb_1}^{min_1} f_{T_L}(t)dt + q_B^n q_A^{[(n+1)b/a]} \right.$$

$$\left. \cdot \int_{min_1}^{(n+1)b_1} f_{T_L}(t)dt \right) , \quad b_1 \leq a_1$$

$$= \sum_{n=0}^{\infty} \left(q_A^n q_B^{[na/b]} \int_{na_1}^{min_2} f_{T_L}(t)dt + q_A^n q_B^{[(n+1)a/b]} \right.$$

$$\left. \cdot \int_{min_2}^{(n+1)a_1} f_{T_L}(t)dt \right. , \quad a_1 \leq b_1$$

where AB_2 = Event time runs out before a kill, and where

$$\min_1 = \min \{(n+1)b_1, [nb/a]a_1 + a_1\} \quad ,$$

and

$$\min_2 = \min \{(n+1)a_1, [na/b]b_1 + b_1\} \quad .$$

$$P(AB) = P(AB_1) + P(AB_2) \quad .$$

$$P(A) \, P(T_D = na_1 | A) = p_A \, q_A^{n-1} q_B^{[na/b]} \int_{na_1}^{\infty} f_{T_L}(t) dt$$

$$P(AB_1) \, P(T_D = nba_1 | AB_1) = \frac{p_A p_B}{q_A q_B} \, (q_A^b q_B^a)^n \int_{nba_1}^{\infty} f_{T_L}(t) dt$$

$$P(AB_2) \, g_{AB_2}(t) = q_A^{[t/a_1]} \, q_B^{[t/b_1]} \, f_{T_L}(t)$$

where

$g_{AB_2}(t) = $ pdf of time to end of duel, given time runs out before a kill.

The two cases are kept distinct because the first is a pmf and the second is a pdf.

A & G2

B. FIXED TIME LIMIT

Let $T_L = \tau$ (a constant).

$$P(A) = \begin{cases} P_A \displaystyle\sum_{n=1}^{[\tau/a_1]} q_A^{n-1} q_B^{[na/b]} \;, & \tau \geq a_1 \\[3em] 0 \;, & \tau < a_1 \end{cases}$$

Simultaneous Kills

$$P(AB_1) = \begin{cases} \dfrac{P_A P_B \, q_A^{b-1} q_B^{a-1}}{1 - q_A^{b} q_B^{a}} \left\{ 1 - (q_A^{b} q_B^{a})^{[\tau/a_1 b]} \right\}, & \tau \geq ab_1 = ba_1 \\[3em] 0 \;, & \tau < ab_1 = ba_1 \end{cases}$$

Time Runs Out Before a Kill

$$P(AB_2) = q_A^{[\tau/a_1]} \, q_B^{[\tau/b_1]} \qquad , \qquad 0 < \tau < \infty$$

A2 $\qquad P(AB) = P(AB_1) + P(AB_2)$

$$P(A)\, P(T_D = na_1 | A) = P_A \, q_A^{n-1} q_B^{[na/b]} \qquad , \qquad n = 1, 2, \ldots, [\tau/a_1]$$

$$P(AB_1)\, P(T_D = nba_1 | AB_1) = \frac{P_A P_B}{q_A q_B} (q_A^{b} q_B^{a})^{n} \quad , \quad \begin{cases} n = 1, 2, \ldots, [\tau/ab_1] \\[1em] = [\tau/ba_1] \end{cases}$$

$$P(AB_2) \, g_{AB_2}(t) = q_A^{[t/a_1]} \, q_B^{[t/b_1]} \, \delta(t - \tau)$$

$$g(t) = P(A) \, P(T_D = na_1 | A) \, \delta(t - na_1) + P(B) \, P(T_D = nb_1 | B) \, \delta(t - nb_1)$$

$$+ P(AB_1) \, P(T_D = nba_1 | AB_1) \, \delta(t - nba_1)$$
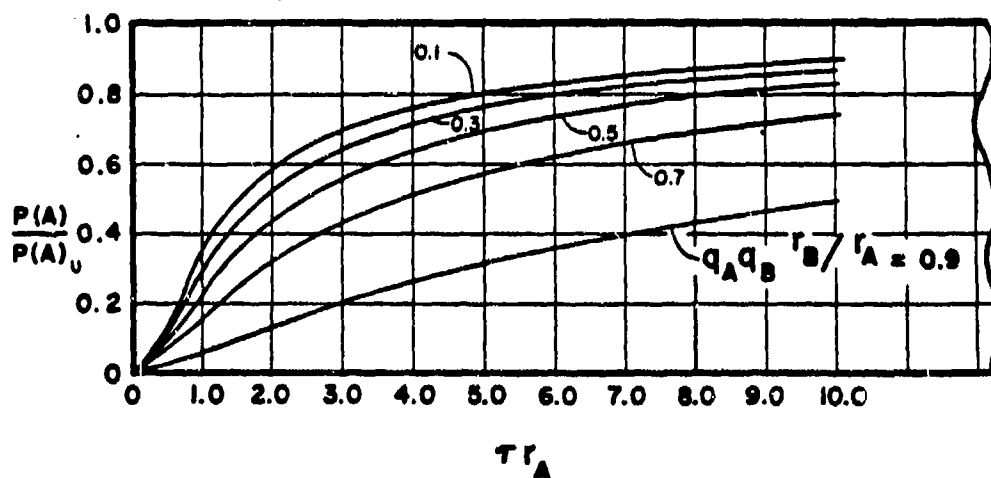
$$+ P(AB_2) \, g_{AB_2}(t) \ . \qquad \text{A \& G2}$$

<u>Example 1</u>: Let $a_1 = cb_1$, or $a = c$ and $b = 1$, where c is a positive integer and $f_{T_L} = (1/\tau)e^{-t/\tau}$.

$$P(A) \, P(T_D = na_1 | A) = \frac{P_A}{q_A} \, (q_A q_B^c \exp(-a_1/\tau))^n \ , \qquad \text{A \& G2}$$

$$P(A) = \frac{P_A \, q_B^c}{\exp(a_1/\tau) - q_A q_B^c} \ , \qquad \text{A2}$$

$$P(AB_1) \, P(T_{D_1} = na_1 | AB_1) = \frac{P_A P_B}{q_A q_B} \, (q_A q_B^c \exp(-a_1/\tau))^n \ , \qquad \text{A \& G2}$$

$$P(AB_1) = \frac{P_A P_B \, q_B^{c-1}}{\exp(a_1/\tau) - q_A q_B^c} \ , \qquad \text{A2}$$

$$P(AB_2) \, g_{AB_2}(t) = \frac{q_A^{[t/a_1]} \, q_B^{[tc/a_1]}}{\tau} \, e^{-t/\tau} \ , \qquad \text{A \& G2}$$

$$P(AB_2) = \frac{\{1 - \exp(-b_1/\tau)\} \, \{1 - (q_B \exp(-b_1/\tau))^c\}}{\{1 - q_B \exp(-b_1/\tau)\}\{1 - q_A(q_B \exp(-b_1/\tau))^c\}} \ . \qquad \text{A2}$$

<u>Example 2</u>: Let $a_1 = cb_1$, or $a = c$ and $b = 1$, where c is a positive integer, and $f_{T_L}(t) = (1/\tau)e^{-t/\tau}$ (see Example 1). Also, let $P(A)_U$ be the outcome of the corresponding unlimited FD, thus,

$$P(A)_U = \frac{P_A q_B^c}{1 - q_A q_B^c} \qquad \text{and} \qquad \frac{P(A)}{P(A)_U} = \frac{1 - q_A q_B^c}{\exp(a_1/\tau) - q_A q_B^c} .$$

This may be rewritten in previously used terms by observing that $a_1 = 1/r_A$ and $c = a_1/b_1 = r_B/r_A$, where r_A and r_B are rates of fire. Thus,

$$\frac{P(A)}{P(A)_U} = \frac{1 - q_A q_B^{r_A/r_B}}{\exp[1/\tau r_A] - q_A q_B^{r_B/r_A}} .$$

A2

Example 3: Let $a_1 = cb_1$ (c is a positive integer), from which $a = c$, $b = 1$, and $T_L = \tau$ (a constant).

$$P(A) \ P(T_D = na_1 | A) = \frac{p_A}{q_A} (q_A q_B^c)^n \qquad\qquad , \ n = 1, 2, \ldots, [\tau/a_1] \qquad \text{A \& G2}$$

$$P(A) = p_A \ q_B^c \left\{ \frac{1 - (q_A q_B^c)^{[\tau/a_1]}}{1 - q_A q_B^c} \right\} \qquad , \ \tau \geq a_1$$

$$= 0 \qquad\qquad , \ \tau < a_1 \qquad \text{A2}$$

$$P(AB_1) \ P(T_D = nba_1 | AB_1) = \frac{p_A p_B}{q_A q_B} (q_A q_B^c)^n \qquad , \ n = 1, 2, \ldots, [\tau/a_1] \qquad \text{A \& G2}$$

$$P(AB_1) = \frac{p_A p_B \ q_B^{c-1}}{1 - q_A q_B^c} \left\{ 1 - (q_A q_B^c)^{[\tau/a_1]} \right\} \qquad , \ \tau \geq a_1$$

$$= 0 \qquad\qquad , \ \tau < a_1 \qquad \text{A2}$$

$$P(AB_2) \ g_{AB_2}(t) = q_A^{[t/a_1]} \ q_B^{[tc/a_1]} \ \delta(t - \tau) \qquad , \qquad\qquad \text{A \& G2}$$

$$P(AB_2) = q_A^{[\tau/a_1]} \ q_B^{[\tau c/a_1]} \qquad\qquad . \qquad \text{A2}$$

Example 4: Let $a_1 = cb_1$, or $a = c$ and $b = 1$, where c is a positive integer and $T_L = \tau$ (a constant). See Example 3. Also let $P(A)_U$ be the outcome of the corresponding unlimited FD; thus,

$$P(A)_U = \frac{P_A q_B^c}{1 - q_A q_B^c}$$

and

$$\frac{P(A)}{P(A)_U} = 1 - (q_A q_B^c)^{[\tau/a_1]} = 1 - (q_A q_B^{r_B/r_A})^{[\tau r_A]}$$

where, as before, $a_1 = 1/r_A$ and $c = r_B/r_A$.

$$\tau r_A$$
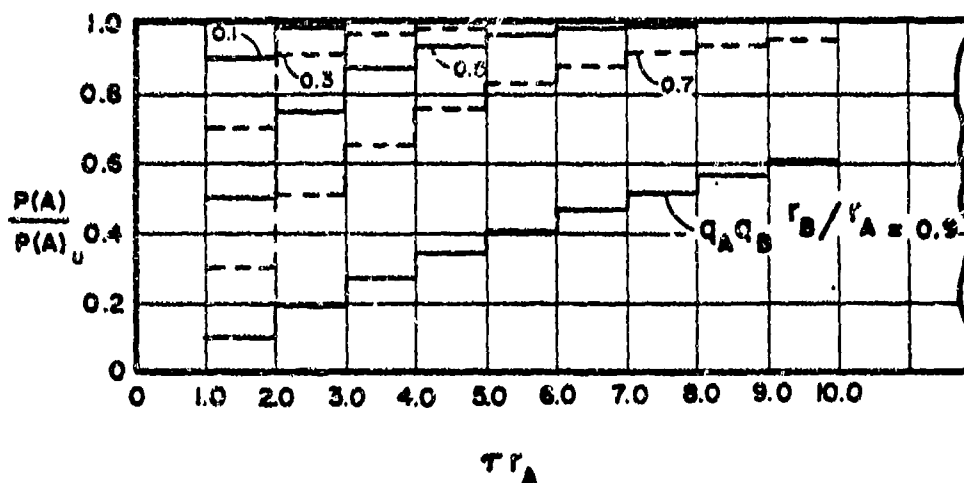
A2

VI. INTERRUPTED FIRING - DISPLACEMENTS

A. $\underline{X_A = X_B = C \quad (\text{SOME CONSTANT})}$

Simultaneous firing

P_i = hit probability for i

q_i = miss probability for i $\bigg\}$ for $i = A, B$

s_i = "near miss" probability for i; $p_i + q_i + s_i = 1$, $i = A, B$. A "near miss" causes opponent to displace and miss one firing turn. A contestant is vulnerable during displacement.

$$P(A) = \frac{p_A(1 - s_B)(1 - s_A s_B - p_B)}{(1 - s_A s_B)[p_A(1 - s_B) + p_B(1 - s_A) - p_A p_B]} \quad ,$$

$$P(AB) = \frac{p_A p_B(1 - s_A)(1 - s_B)}{(1 - s_A s_B)[p_A(1 - s_B) + p_B(1 - s_A) - p_A p_B]} \quad .$$

Example: Let

$$\rho_A = \frac{p_A}{1 - s_A} \; ; \qquad \rho_B = \frac{p_B}{1 - s_B} \; ; \qquad \text{s} = \frac{1 - s_A s_B}{1 - s_B} \; ;$$

then

$$P(A) = \frac{\rho_A(\text{s} - \rho_B)}{\text{s}(\rho_A + \rho_B - \rho_A \rho_B)} \, , \qquad 0 \le \rho_A, \quad \rho_B \le 1 \text{ and } \text{s} \ge 1.$$

(See next page for curves.) N.b., for Fig. (d), i.e., $\text{s} = \infty$, the curves are only limits as $\text{s} \longrightarrow \infty$, the solutions are actually the points on $\rho_B = 1$.

B. $\underline{X_A = X_B = c \text{ (A CONSTANT)}, t_s < c}$

This is the case of alternate firing with A firing first. Let

p_i = hit probability, $i = A, B$

(a)

(b)

(c)

(d)

A & W1

The Duel with Displacements
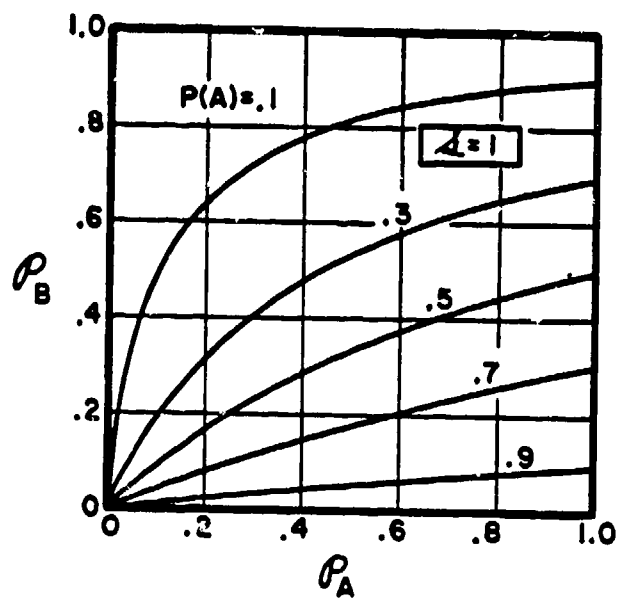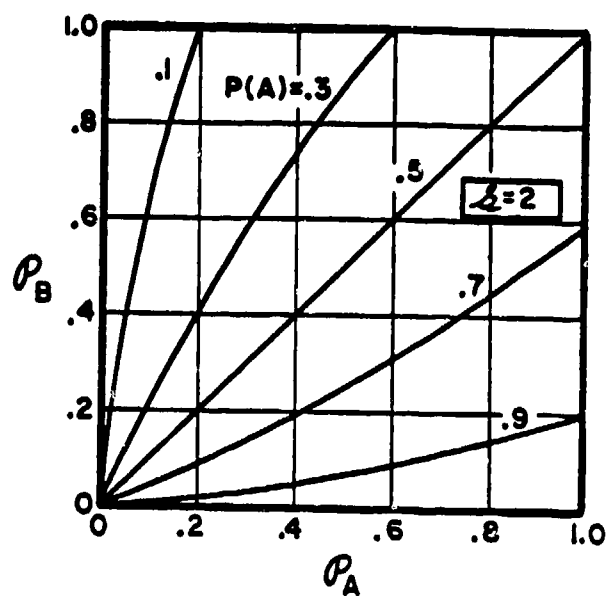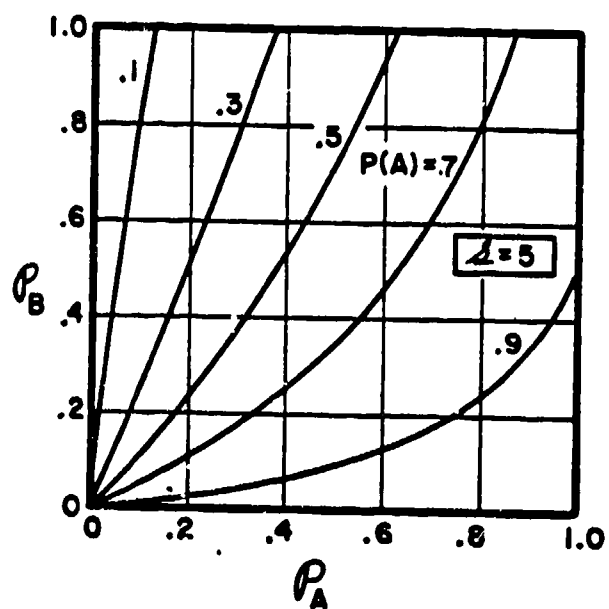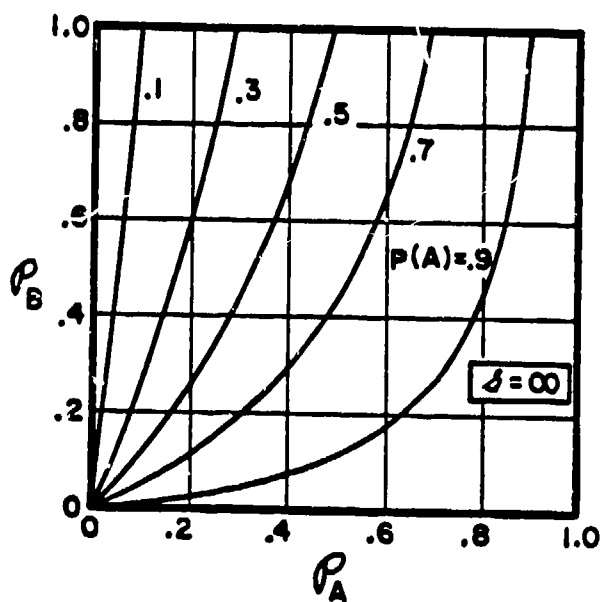
q_i = miss probability,

s_i = "near miss" probability, $p_i + q_i + s_i = 1$, where a "near miss" causes opponent to displace and miss one firing time. A contestant is vulnerable during displacement, $i = A,B$.

If, N_i = number of rounds fired by $i = A,B$ to a kill, then

$$G_{N_A}(z) = \frac{p_A[s_A s_B + q_A - q_A q_B z]}{1 - (q_B + q_A + r_A r_B)z + q_A q_B z^2}$$

and

$$P_{N_A}(n) = \left. \frac{d^{n-2}}{dz^{n-2}} (G_{N_A}(z)) \right|_{z=0} \quad , \quad n = 2,3,\ldots$$

$$P_{N_A}(1) = p_A$$

$$G_{N_B}(z) = \frac{p_B\, r_B}{1 - (q_A + q_B + r_A r_B)z + q_A q_B z^2}$$

and

$$P_{N_B}(n) = \left. \frac{d^{n-1}}{dz^{n-1}} (G_{N_B}(z)) \right|_{z=0} \quad , \quad n = 1,2,\ldots$$

then

$$P[A] = \frac{p_A(1 - q_B)}{1 - (q_A + q_B + s_A s_B) + q_A q_B} \quad ,$$

Sal
$$P(B) = \frac{p_B s_A}{1 - (q_A + q_B + s_A s_B) + q_A q_B} \quad .$$

VII. TIME-OF-FLIGHT INCLUDED

A. NO DELAY

Each contestant fires as rapidly as possible (i.e., does _not_ wait for round in air to land before preparing and firing next round). Let

$$X_A = a_1 \,, \quad X_B = b_1$$
$$T_F = \tau_A \text{ (fixed)}, \quad T_F = \tau_B \text{ (fixed)}.$$

Let

$[x]$ - means the largest integer less than or equal to x ,

$\langle x \rangle$ - means maximum of the largest integer less than x or zero.

$$P(A) = p_A \sum_{j=1}^{\infty} q_A^{j-1} q_B^{[(ja/b) + (\tau_A/b_1)]}$$

$$P(B) = p_B \sum_{j=1}^{\infty} q_B^{j-1} q_A^{[(jb/a) + (\tau_B/a_1)]}$$

$$P(AB) = p_A \sum_{j=1}^{\infty} q_A^{j-1} \left(q_B^{\langle (ja/b) - (\tau_B/b_1) \rangle} - q_B^{[(ja/b) + (\tau_A/b_1)]} \right) \quad .$$

Example: Let $a_1/b_1 = c$, i.e., $a = c$, $b = 1$, where c is a positive integer. Then,

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$$P(A) = \frac{p_A\, q_B^{[\tau_A/b_1]+c}}{1 - q_A q_B^c}$$

$$P(AB) = 1 - q_A^{[\alpha/c]}\left\{ 1 - \frac{p_A\, q_B^{c-\alpha+[\alpha/c]c}}{1 - q_A q_B^c} - \frac{p_A\, q_B^{[\tau_A/b_1]+c}}{1 - q_A q_B^c} \right\}$$

where

$$\alpha = [\tau_B/b_1] + 1 \ .$$

B. DUEL WITH DELAY

Each contestant waits until his last round has landed before he prepares and fires his next round.

Let $X_A = a_1$; $X_B = b_1$; $T_{FA} = \tau_A$; $T_{FB} = \tau_B$ (T_{FA}, T_{FB} - fixed). The results are the same as the no delay case, A. above, except replace a_1 by $a_1 + \tau_A$ and b_1 by $b_1 + \tau_B$, and let a/b be the reduced ratio of $(a_1 + \tau_A)/(b_1 + \tau_B)$, if the numerator and denominator contain a common factor.

A3

VIII. MARKOV-DEPENDENT FIRE

FOR A

E_j - states of B, $j = 0, 1, \ldots, m$

E_0 - A has B as a target (starting state at time zero)

E_m - A has killed B (absorbing state)

E_i - other specified arbitrary states, $i \neq 0,m$

p_{ij} - P[B goes from state E_i to E_j], transition probabilities

$m_A^T = (1,0,\ldots,0)$ - m components .

Then:

$$
S_A = \left(
\begin{array}{cccc|c}
p_{00} & p_{01} & \cdots & p_{0,m-1} & p_{0,m} \\
p_{10} & p_{11} & \cdots & p_{1,m-1} & p_{1,m} \\
\vdots & \vdots & & \vdots & \vdots \\
p_{m-1,0} & p_{m-1,1} & \cdots & p_{m-1,m-1} & p_{m-1,m} \\
\hline
0 & 0 & \cdots & 0 & 1
\end{array}
\right)
$$

$$
= \left(
\begin{array}{c|c}
P_A & t_A \\
\hline
0 & 1
\end{array}
\right) \quad , \quad \text{transition matrix} .
$$

FOR B

Replace E by F and interchange m and ℓ, A and B, and a and b; n.b., S_B will have <u>different</u> transition probabilities.

A. <u>FD - FIFT</u>

$$P[A \; ; N_A = n] = \underline{m}_A^T \ \underline{P}_A^{n-1} \ \underline{t}_A \ \underline{m}_B^T \ \underline{P}_B^{[na/b]} \ \underline{e}_\ell$$

$$P[A] = \underline{m}_A^T \times \underline{e}_\ell^T \Big[\underline{I} - \underline{P}_A^b \times (\underline{P}_B^a)^T \Big]^{-1} \Big(\sum_{i=1}^{b} \underline{P}_A^{i-1} \ \underline{t}_A \ \underline{m}_B^T \ \underline{P}_B^{[ia/b]} \Big)^\nu$$

$$P[B] = \underline{m}_B^T \times \underline{e}_m^T \Big[\underline{I} - \underline{P}_B^a \times (\underline{P}_A^b)^T \Big]^{-1} \Big(\sum_{j=1}^{a} \underline{P}_B^{j-1} \ \underline{t}_B \ \underline{m}_A^T \ \underline{P}_A^{[jb/a]} \Big)^\nu$$

$$P[AB] = 1 - P[A] - P[B] \; .$$

B. <u>INITIAL SURPRISE BY A</u>

1. <u>A Fires y (a Fixed Number) Rounds Before B is Alerted and FD Begins, y = 1,2,...</u>

$$P[A \, , N_A = n] = \underline{m}_A^T \ \underline{P}_A^{n-1} \ \underline{t}_A \qquad\qquad ; \ n \leq y, \ N_B = 0$$

$$= \underline{m}_A^T \ \underline{P}_A^{n-1} \ \underline{t}_A \ \underline{m}_B^T \ \underline{P}_B^{[(n-y)(a/b)]} \ \underline{e}_\ell ; \ n > y \, , \ N_B = [(n-y)(a/b)]$$

$$P[B \, , N_B = n] = \underline{m}_B^T \ \underline{P}_B^{n-1} \ \underline{t}_B \ \underline{m}_A^T \ \underline{P}_A \underline{P}_A^{[nb/a]} \ \underline{e}_m \ ; \ n = 1,2,\ldots, \ N_A = y + [nb/a]$$

$$P(A) = \sum_{n=1}^{\infty} P[A \, , N_A = n]$$

$$= 1 - \underline{m}_A^T \ \underline{P}_A^y \Big\{ \underline{e}_m - \underline{I} \times \underline{e}_\ell^T [\underline{I} - \underline{P}_A^b \times (\underline{P}_B^a)^T]^{-1} \quad .$$

$$\cdot \left(\sum_{i=1}^{\hat{\lambda}} P_A^{i-1} \, t_A \, m_B^T \, P_B^{[ia/b]} \right)^\nu \Big\}$$

$$P(B) \;=\; \sum_{n=1}^{\infty} P[B, N_B = n]$$

$$=\; m_B^T \times [P_A^n \, e_m]^T \, [I - P_B^a \times (P_A^b)^T]^{-1}$$

$$\cdot \left(\sum_{j=1}^{a} P_B^{j-1} \, t_B \, m_A^T \, P_A^{[jb/a]} \right)^\nu \quad .$$

$$P(AB) \;=\; 1 - P(A) - P(B) \quad .$$

2. <u>A Fires Y (a rv) Rounds Before B Starts FD</u>

$$p_Y(y) = \begin{cases} P_a \, q_a^{y-1}; & y = 1, 2, \ldots, \quad q_a = 1 - P_a \\[2mm] 0 & , \qquad \text{elsewhere} \end{cases} \quad ,$$

where

p_a = the probability that B acquires A on each unanswered round fixed by A.

$$P(A) \;=\; 1 - P_a \, m_A^T \, P_A (I - q_a \, P_A)^{-1}$$

$$\cdot \left[(I - P_A)^{-1} \, t_A - \sum_{i=1}^{\infty} P_A^{i-1} \, t_A \, m_B^T \, P_B^{[ia/b]} \, (I - P_B)^{-1} \, t_B \right] \;=$$

$$= 1 - P_a \; \underset{\sim}{m}_A^T \; \underset{\sim}{P}_A (\underset{\sim}{I} - q_a \; \underset{\sim}{P}_A)^{-1} \left[\underset{\sim}{e}_m - \underset{\sim}{I} \times \underset{\sim}{e}_\ell^T \{ \underset{\sim}{I} - \underset{\sim}{P}_A^b \times (\underset{\sim}{P}_B^a) \}^{-1} \right.$$

$$\left. \cdot \left(\sum_{i=1}^{b} \underset{\sim}{P}_A^{i-1} \; \underset{\sim}{t}_A \; \underset{\sim}{m}_b^T \; \underset{\sim}{P}_B^{[ia/b]} \right)^\nu \right]$$

$$P(B) = P_a \; \underset{\sim}{m}_B^T \left(\sum_{i=1}^{\infty} \underset{\sim}{P}_B^{i-1} \; \underset{\sim}{t}_B \; \underset{\sim}{m}_A^T \; \underset{\sim}{P}_A^{[ib/a]} \right) \underset{\sim}{P}_A (\underset{\sim}{I} - q_a \; \underset{\sim}{P}_A)^{-1} (\underset{\sim}{I} - \underset{\sim}{P}_A)^{-1} \; \underset{\sim}{t}_A$$

$$= P_a \; \underset{\sim}{m}_B^T \times [\underset{\sim}{P}_A (\underset{\sim}{I} - q_a \; \underset{\sim}{P}_A)^{-1} \; \underset{\sim}{e}_m \; [\underset{\sim}{I} - \underset{\sim}{P}_B^a \times (\underset{\sim}{P}_A^b)^T]^{-1}$$

$$\cdot \left(\sum_{i=1}^{a} \underset{\sim}{P}_B^{i-1} \; \underset{\sim}{t}_B \; \underset{\sim}{m}_A^T \; \underset{\sim}{P}_A^{[ib/a]} \right)^\nu$$

$$P(AB) = 1 - P(A) - P(B) .$$

<div align="right">Ba2</div>

Note: In 1. and 2., the case where B has initial surprise may be

obtained by interchanging A and B, a and b, and

ℓ and m.

C. <u>BURST FIRING</u>

Let

$$A - \begin{cases} z \text{ (fixed) rounds in a burst} \\ a \text{ time units between rounds in a burst} \\ o \text{ time units between bursts} \end{cases}$$

$$B - \quad b \text{ time units between rounds (no bursts)}$$

Then

$$f(i) \equiv f(i;a,b,z,\rho) \triangleq \left[\frac{ia + \left[\frac{i-1}{z}\right](\rho - a)}{b} \right]$$

is the number of rounds fired by B, while A is firing i rounds, and

$$g(j) \equiv g(j;a,b,z,\rho) \triangleq \left[\frac{jb - (\rho - a)\left[\frac{jb - a}{za + \rho - a}\right]}{a} \right]$$

$$\cdot \left\{ \left[\frac{jb - a}{za + \rho - a}\right] - \left[\frac{jb - za}{za + \rho - a}\right] \right\} + z\left[\frac{jb + \rho - a}{za + \rho - a}\right]$$

$$\cdot \left\{ \left[\frac{jb - za}{za + \rho - a}\right] - \left[\frac{jb - za - \rho}{za + \rho - a}\right] \right\}$$

is the number of rounds fired by A while B is firing j rounds, where [x] is the largest integer $\leq$ x.

1. <u>Initial Condition Both Start at Time Zero; A Waits a Time Units to Fire First Round; B Waits b Time Units to Fire First Round</u>

$$P[A, N_A = n] = \underset{\sim}{m_A^T} \; \underset{\sim}{P_A^{n-1}} \; \underset{\sim}{t_A} \; \underset{\sim}{m_P^T} \; \underset{\sim}{P_B^{f(n)}} \; \underset{\sim}{e_\ell}$$

$$P[B, N_B = n] = \underset{\sim}{m_B^T} \; \underset{\sim}{P_B^{n-1}} \; \underset{\sim}{t_B} \; \underset{\sim}{m_P^T} \; \underset{\sim}{P_A^{g(n)}} \; \underset{\sim}{e_m}$$

$$P(A) = \sum_{i=1}^{\infty} \underset{\sim}{m_A^T} \underset{\sim}{P_A^{i-1}} \underset{\sim}{m_B^T} \underset{\sim}{P_B^{f(i)}} \underset{\sim}{e_\ell}$$

$$P(B) = \sum_{j=1}^{\infty} \underset{\sim}{m_B^T} \underset{\sim}{P_B^{j-1}} \underset{\sim}{t_B} \underset{\sim}{m_A^T} \underset{\sim}{P_A^{g(j)}} \underset{\sim}{e_m}$$

$$P(AB) = 1 - P(A) - P(B)$$

where

$P(A)$ may be approximated to an error $< \epsilon_1$ by truncating the infinite sum at N_1

$P(B)$ may be approximated to an error $< \epsilon_2$ by truncation at N_2

and where

$$N_1 = \min \{ i : \underset{\sim}{m_B^T} \underset{\sim}{P_B^{f(i+1)}} \underset{\sim}{e_\ell} \underset{\sim}{m_A^T} \underset{\sim}{P_A^i} \underset{\sim}{e_m} < \epsilon_1 \} \ ,$$

$$N_2 = \min \{ j : \underset{\sim}{m_A^T} \underset{\sim}{P_A^{g(j+1)}} \underset{\sim}{e_m} \underset{\sim}{m_B^T} \underset{\sim}{P_B^j} \underset{\sim}{e_\ell} < \epsilon_2 \} \ .$$

2. <u>Initial Conditions Are A Fires y (Fixed Number) Rounds Before B Begins (Initial Surprise)</u>

$$P[A, N_A = n] = \underset{\sim}{m_A^T} \underset{\sim}{P_A^{n-1}} \underset{\sim}{t_A} \qquad\qquad , \qquad n \le y$$

$$= \underset{\sim}{m_A^T} \underset{\sim}{P_A^{n-1}} \underset{\sim}{t_A} \underset{\sim}{m_B^T} \underset{\sim}{P_B^{f(n-y)}} \underset{\sim}{e_\ell} \ , \qquad n > y$$

$$P[B, N_B = n] = \underset{\sim}{m_B^T} \underset{\sim}{P_B^{n-1}} \underset{\sim}{t_B} \underset{\sim}{m_A^T} \underset{\sim}{P_A^y} \underset{\sim}{P_A^{g(n)}} \underset{\sim}{e_m}$$

$$P(A) = 1 - \underset{\sim}{m_A^T} \underset{\sim}{P_A^y} \underset{\sim}{e_m} + \sum_{i=1}^{\infty} \underset{\sim}{m_A^T} \underset{\sim}{P_A^y} \underset{\sim}{P_A^{i-1}} \underset{\sim}{t_A} \underset{\sim}{m_B^T} \underset{\sim}{P_B^{f(i)}} \underset{\sim}{e_\ell}$$

$$P(B) = \sum_{j=1}^{\infty} \underset{\sim}{m_B^T} \underset{\sim}{P_B^{j-1}} \underset{\sim}{t_B} \underset{\sim}{m_A^T} \underset{\sim}{P_A^n} \underset{\sim}{P_A^{g(j)}} \underset{\sim}{e_m}$$

$$P(AB) = 1 - P(A) - P(B)$$

while the results for initial surprise by B are obtained by interchanging A and B, ℓ and m, and $f(i)$ and $g(j)$.

3. <u>Initial Conditions Are A Fires Y (A RV) Rounds Before B Begins (Random Surprise)</u>

Let

$$p_Y(y) = \begin{cases} p_a \, q_a^{y-1}; \; q_a = 1 - p_a, & y = 1,2,\dots \\ \\ 0 & , \quad \text{elsewhere} \end{cases}$$

and

p_a = probability that B acquires A on each unanswered round fired by A .

Then

$$P(A) = 1 - p_a \, \underset{\sim}{m_A^T} \, \underset{\sim}{P_A} (\underset{\sim}{I} - q_a \, \underset{\sim}{P_A})^{-1} \, \underset{\sim}{e_m} + p_a \, \underset{\sim}{m_A^T} \, \underset{\sim}{P_A} (\underset{\sim}{I} - q_a \, \underset{\sim}{P_A})^{-1}$$

$$\cdot \sum_{i=1}^{\infty} \underset{\sim}{P_A^{i-1}} \, \underset{\sim}{t_A} \, \underset{\sim}{m_B^T} \, \underset{\sim}{P_B^{f(i)}} \, \underset{\sim}{e_\ell}$$

$$P(B) = p_a \, \underset{\sim}{m_B^T} \left(\sum_{j=1}^{\infty} \underset{\sim}{P_B^{j-1}} \, \underset{\sim}{t_B} \, \underset{\sim}{m_A^T} \, \underset{\sim}{P_A^{g(j)}} \right) \underset{\sim}{P_A} (\underset{\sim}{I} - q_a \, \underset{\sim}{P_A})^{-1} \, \underset{\sim}{e_m}$$

$P(AB) = 1 - P(A) - P(B).$

Results for random initial surprise by B are obtained by interchanging A and B, a and b, ℓ and m, and $f(i)$ and $g(j)$.

Approximations, to any desired degree of accuracy, for all these probabilities (in C2 and C3 above) may be obtained by taking partial sums where infinite sums are given. For example, an appropriate stopping rule for $P(B)$ in C3 is:

stop at $N = \min \{ j : p_a \, m_B^T \, P_B^j \, e_\ell \, m_A^T \, P_A^{1+g(j+1)} (I - q_a \, P_A)^{-1} \, e_m < \varepsilon \}$,

where ε is the desired bound on the error.

FUNDAMENTAL DUEL - CONTINUOUS RANDOM
INTERFIRING TIMES
(FD - CRIFT)

I. FD - CRIFT

X_A, X_B Are rv's

$$P(A) = \frac{1}{2} + \frac{1}{2\pi i}(P)\int_{-\infty}^{\infty} \Phi_A(-u)\,\Phi_B(u)\,\frac{du}{u}$$

$$= \frac{1}{2\pi i}\int_L \Phi_A(-u)\,\Phi_B(u)\,\frac{du}{u}$$

$$= 1 + \frac{1}{2\pi i}\int_U \Phi_A(-u)\,\Phi_B(u)\,\frac{du}{u}$$

where

$$\Phi_A(-u) = \frac{p_A\,\phi_A(-u)}{1 - q_A\,\phi_A(-u)} \qquad \text{and} \qquad \Phi_B(u) = \frac{p_B\,\phi_B(u)}{1 - q_B\,\phi_B(u)}$$

$$P(B) = 1 - P(A) \hspace{4cm} \text{W \& Al}$$

$$P(A)\,g_A(t) = \frac{1}{4\pi^2 i}\left\{\int_{-\infty}^{\infty} e^{-iwt}\,\Phi_A(w)dw\right\}\left\{\int_{-\infty}^{\infty}\frac{e^{-iut}[\Phi_B(u) - 1]du}{u}\right\}$$

$$= \frac{1}{4\pi^2 i}\left\{\int_{-\infty}^{\infty} e^{-iwt}\,\Phi_A(w)dw\right\}\left\{\int_L \frac{e^{-iut}\,\Phi_B(u)du}{u}\right\}$$

$$P(A)\,\psi_A(u) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\Phi_A(u - w)[\Phi_B(w) - 1]dw}{w}$$

$$= \frac{1}{2\pi i}\int_L \frac{\Phi_A(u - w)\,\Phi_B(w)dw}{w}$$

$$P(A)\, \mu_n(A) = \frac{1}{2\pi i^{n+1}} \int_{-\infty}^{\infty} \frac{\Phi_A^{(n)}(-w)[\Phi_B(w) - 1]dw}{w}$$

$$= \frac{1}{2\pi i^{n+1}} \int_L \frac{\Phi_A^{(n)}(-w)\, \Phi_B(w)dw}{w}$$

A & G2 $$g(t) = g_A(t)\, P(A) + g_B(t)\, P(B) = h_A(t)\, H_B^c(t) + h_B(t)\, H_A^c(t)$$

$$P[N_A = n | A] = \frac{p_A\, q_A^{n-1}}{P(A)} \left[\frac{1}{2} - \frac{1}{2\pi i}\, (P) \int_{-\infty}^{\infty} \frac{\phi_A^n(u)\, \Phi_B(-u)du}{u} \right]$$

$$= \frac{p_A\, q_A^{n-1}}{P(A)} \left[1 - \frac{1}{2\pi i} \int_L \frac{\phi_A^n(u)\, \Phi_B(-u)du}{u} \right]$$

$$= \frac{-p_A\, q_A^{n-1}}{P(A)}\, \frac{1}{2\pi i} \int_U \frac{\phi_A^n(u)\, \Phi_B(-u)du}{u} \quad , \qquad n \geq 1$$

$$E[N_A | A] = \frac{1}{P(A)} \left[\frac{1}{2p_A} - \frac{1}{2\pi i}\, (P) \int_{-\infty}^{\infty} \frac{\phi_A(u)\, \Phi_B(-u)du}{[1 - q_A\, \phi_A(u)]^2\, u} \right]$$

$$= \frac{1}{P(A)} \left[\frac{1}{p_A} - \frac{p_A}{2\pi i} \int_L \frac{\phi_A(u)\, \Phi_B(-u)du}{[1 - q_A\, \phi_A(u)]^2\, u} \right]$$

$$= - \frac{p_A}{P(A)\, 2\pi i} \int_U \frac{\phi_A(u)\, \Phi_B(-u)\, du}{[1 - q_A\, \phi_A(u)]^2\, u}$$

$$E[N_A^2 | A] = \frac{1}{P(A)} \left\{ \frac{1 + q_A}{2p_A^2} + \frac{p_A}{2\pi i}\, (P) \int_{-\infty}^{\infty} \frac{\phi_A(u)\, \Phi_B(-u)[1 + q_A\, \phi_A(u)du}{[1 - q_A\, \phi_A(u)]^3\, u} \right\} =$$

$$= \frac{1}{P(A)} \left[\frac{1 + q_A}{p_A^2} - \frac{p_A}{2\pi i} \int_L \frac{\phi_A(u) \, \Phi_B(-u)[1 + q_A \, \phi_A(u)]du}{[1 - q_A \, \phi_A(u)]^3 \, u} \right]$$

$$= \frac{- p_A}{P(A) \, 2\pi i} \int_U \frac{\phi_A(u) \, \Phi_B(-u)[1 + q_A \, \phi_A(u)]du}{[1 - q_A \, \phi_A(u)]^3 \, u}$$

$$P[N_A \geq n_0 | A] = \frac{1}{P(A)} \left\{ \frac{q_A^{n_0 - 1}}{2} - \frac{p_A}{2\pi i} \, (P) \int_{-\infty}^{\infty} \frac{\phi_A(u) \, \Phi_B(-u)[q_A \, \phi_A(u)]^{n_0 - 1}}{[1 - q_A \, \phi_A(u)]u} du \right\}$$

$$= \frac{1}{P(A)} \left\{ q_A^{n_0 - 1} - \frac{p_A}{2\pi i} \int_L \frac{\phi_A(u) \, \Phi_B(-u)[q_A \, \phi_A(u)]^{n_0 - 1}}{[1 - q_A \, \phi_A(u)]u} du \right\}$$

$$= \frac{- p_A}{P(A) \, 2\pi i} \int_U \frac{\phi_A(u) \, \Phi_B(-u)[q_A \, \phi_A(u)]^{n_0 - 1}}{[1 - q_A \, \phi_A(u)]u} du$$

$$P[N_A = n | B] = \frac{1}{P(B)} \left\{ 1 - \frac{P[N_A = 1 | A] \, P[A]}{p_A} \right\} , \quad n = 0$$

$$= \frac{1}{p_A \, P(B)} \left\{ q_A \, P[N_A = n | A] \, P[A] - P[N_A = n+1 | A] \, P[A] \right\} , \quad n \geq 1$$

$$P[N_A = n] = 1 - \frac{P(A)}{p_A} \, P[N_A = 1 | A] \qquad\qquad , \quad n = 0$$

$$= \frac{P(A)}{p_A} \left\{ P[N_A = n | A] - P[N_A = n+1 | A] \right\} \qquad , \quad n \geq 1$$

$$\Delta P[A] = P[N_A \geq i+1 , A] - P[N_A \geq j+1 , A] , \qquad\qquad \text{for } \alpha_\infty = 1 \quad *$$

FD - CRIFT

A & G1 (Note: The above is a marginal increase in $P(A)$ if A's initial fixed
*
supply is increased from i to j, $j \geq i$.)

Approximations

1) $P(A) \approx I_{\frac{k\alpha}{k\alpha + \ell\beta}}(k, \ell)$ (see example 4 following)

where

$$\alpha = p_A r_A, \quad \beta = p_B r_B, \quad k = \frac{1}{p_A r_A \sigma_A^2 + q_A}, \quad \ell = \frac{1}{p_B r_B \sigma_B^2 + q_B},$$

where

$$r_A = E[X_A] \quad \text{and} \quad r_B = E[X_B],$$

$$\sigma_A^2 = V[X_A] \quad \text{and} \quad \sigma_B^2 = V[X_B].$$

W1 2) $P(A) \approx \dfrac{p_A r_A}{p_A r_A + p_B r_B} + \dfrac{1}{2\left(\dfrac{1}{p_A r_A} + \dfrac{1}{p_B r_B}\right)^2}$

$$\cdot \left\{ \frac{(\sigma_A r_A)^2 - 1}{p_A r_A^2} - \frac{(\sigma_B r_B)^2 - 1}{p_B r_B^2} \right\}.$$

Example 1: Let $X_A \sim \text{ned}(r_A)$ and $X_B \sim \text{ned}(r_B)$.

W & A1 $P(A) = \dfrac{p_A r_A}{p_A r_A + p_B r_B}$ (two different plots of this follow)

C110

$$g_A(t) = g_B(t) = (p_A r_A + p_B r_B) e^{-(p_A r_A + p_B r_B)t}$$

$$P[A \text{ is alive at time } t] = \frac{p_A r_A + p_B r_B e^{-(p_A r_A + p_B r_B)t}}{p_A r_A + p_B r_B} \ . \qquad \text{Sc2}$$

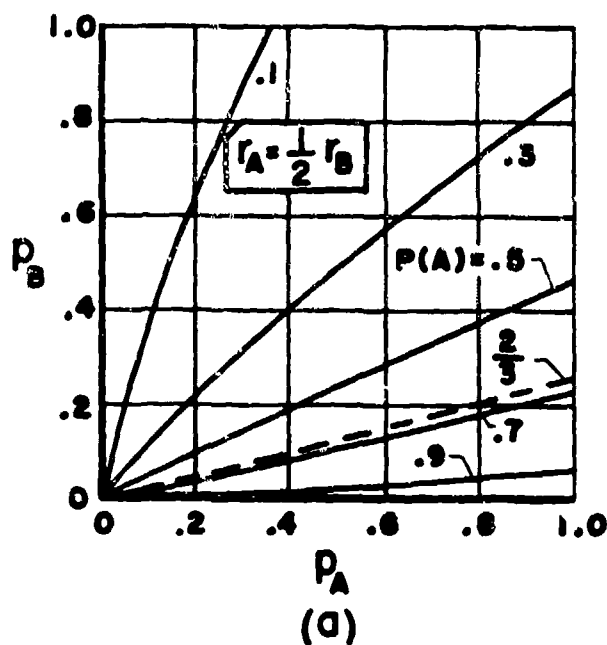
The Fundamental Duel with Negative Exponential Firing Times

C111

(b)

(a)

(c)
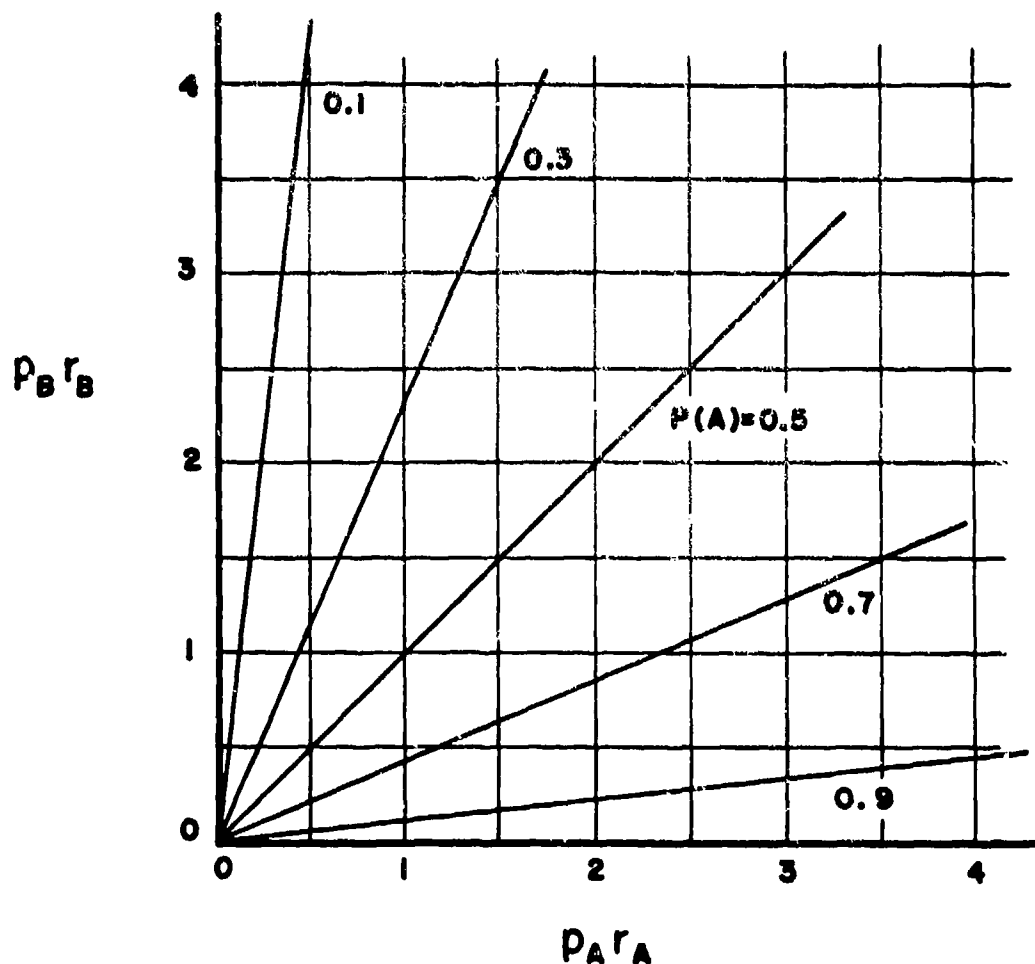
W & A1   The Fundamental Duel with Negative Exponential Firing Times
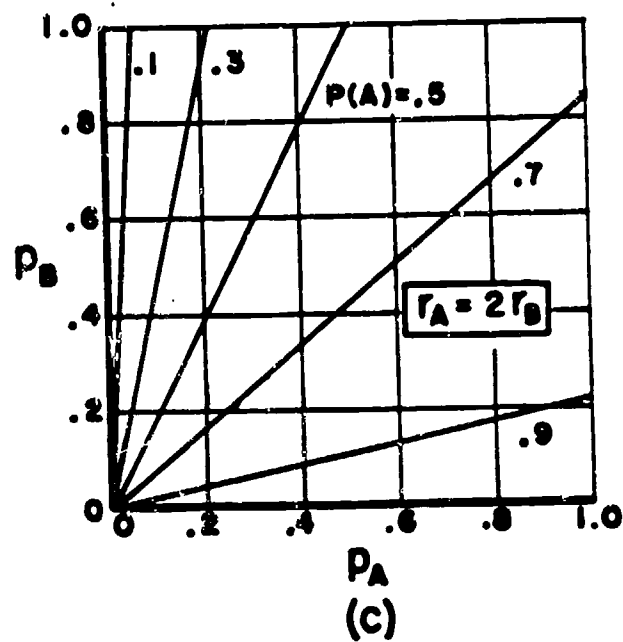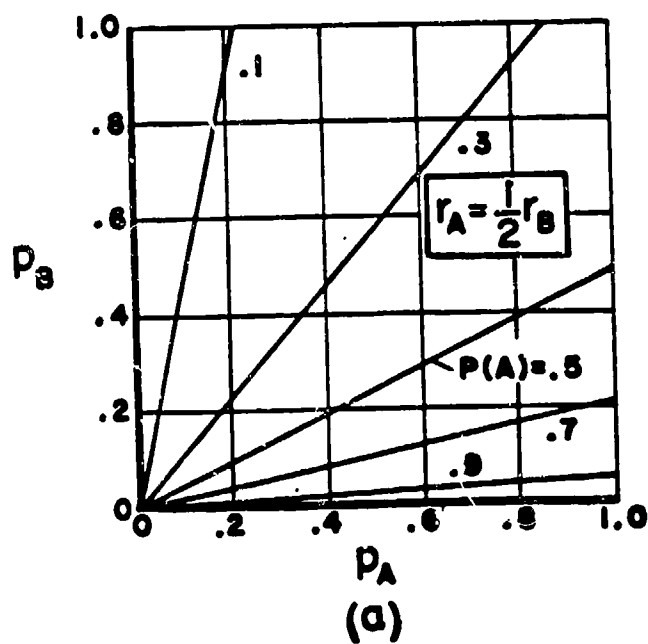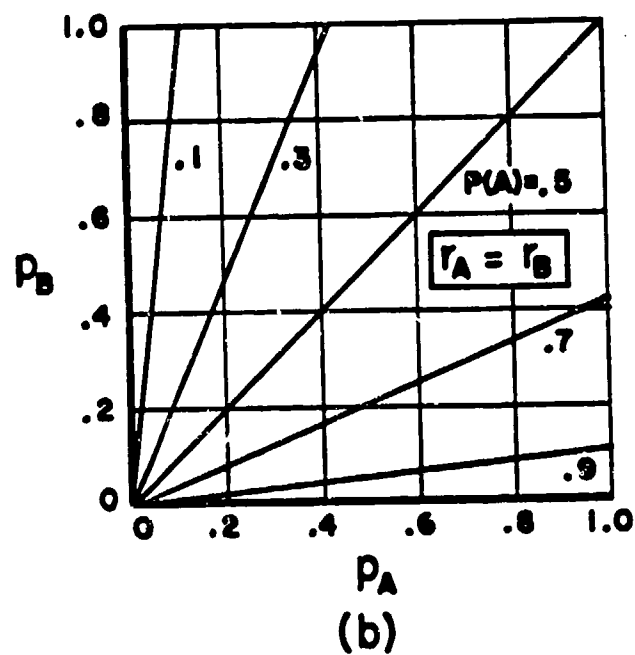
Example 2 : $X_A \sim \text{Erlang}(2, r_A)$ and $X_B \sim \text{Erlang}(2, r_B)$

$$P(A) = p_A r_A^2 \left[\frac{(p_A r_A^2 - p_B r_B^2) + 4 r_B (r_A + r_B)}{(p_A r_A^2 - p_B r_B)^2 + 4 r_A r_B (r_A + r_B)(p_A r_A + p_B r_B)} \right] . \qquad \text{W \& A1}$$

$$P(A)\, g_A(t) = \frac{2 p_A r_A}{\sqrt{q_A q_B}}\; e^{-2(r_A + r_B)t}$$

$$\cdot \sinh 2 r_A \sqrt{q_A}\; t (\sinh 2 r_B \sqrt{q_B}\; t + \sqrt{q_B}\; \cosh 2 r_B\, t) . \qquad \overset{*}{\text{A \& 32}}$$

The Fundamental Duel with Erlang(2) Firing Times   W & A1

(b)

(C)

W & Al
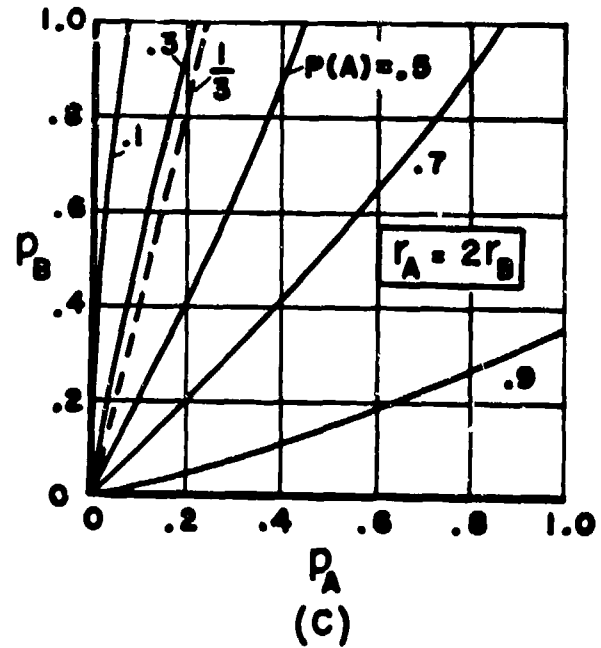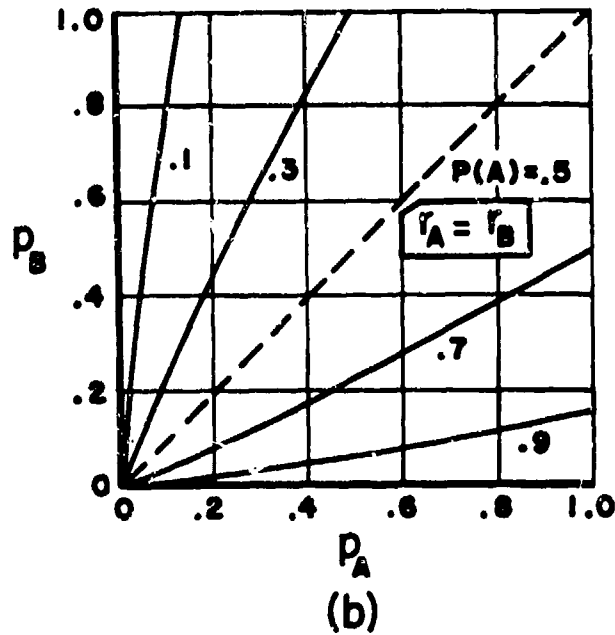
The Fundamental Duel with Erlang(2) Firing Times

Example 3: Let $X_A \sim \text{Erlang}(n, r_A)$ and $X_E \sim \text{Erlang}(m, r_B)$.

$$P(A) = \frac{p_A p_B}{q_B^{(n-1)/m}} \sum_{j=0}^{m-1}$$

$$\cdot \frac{1}{\left\{\left[1 + \frac{mr_B}{nr_A}\left(1 - q_B^{1/m} e^{12\pi j/m}\right)\right]^n - q_A\right\} \left\{\prod_{\substack{k=0 \\ k \neq j}}^{m-1} \left(e^{12\pi j/m} - e^{12\pi k/m}\right)\right\} \left\{1 - q_B^{1/m} e^{12\pi j/m}\right\}}$$

$$P(A) = \frac{P_A P_B}{(2q_B^{1/m})^{m-1}} \sum_{j=0}^{m-1}$$

$$\cdot \frac{(-1)^j \, e^{i2\pi j/m}}{\left\{\left[1 + \frac{mr_B}{nr_A}\left(1 - q_B^{1/m} \, e^{i2\pi j/m}\right)\right]^n - q_A\right\}\left(1 - q_B^{1/m} \, e^{i2\pi j/m}\right) \prod_{\substack{k=0 \\ k \neq j}}^{m-1} \sin \frac{r}{m}\left(k - j\right)}$$

In both expressions, the product term in the denominator is 1 if   A6

$m = 1$.

Example 4: Let $h_A(t) \sim \text{Erlang}(k;\alpha)$ and $h_B(t) \sim \text{Erlang}(\ell;\beta)$.

$$P[A] = I_{\frac{k\alpha}{k\alpha + \ell\beta}} (k, \ell) .$$ W1

Example 5: Let X_A be general and $X_B \sim \text{ned}(r_B)$.

$$P(A) = \frac{P_A \, \phi_A(ir_B \, P_B)}{1 - q_A \, \phi_A(ir_B \, P_B)}$$ *

$$E[N_A , A] = \frac{P_A \, \phi_A(ir_B \, P_B)}{[1 - q_A \, \phi_A(ir_B \, P_B)]^2} .$$ *

Sub-Example: Let $X_A \sim \text{Erlang}(k, r_A)$.

$$P(A) = \frac{P_A}{\left(1 + \frac{r_B}{kr_A} \, P_B \right)^k - q_A}$$

H1
$$E[N_A, A] = \frac{P_A \left(1 + \frac{r_B}{kr_A} P_B \right)^k}{\left[\left(1 + \frac{r_B}{kr_A} P_B \right)^k - q_A \right]^2} .$$

II. VARIATIONS OF INITIAL CONDITIONS

Let $P(A)_f = P(A)$ for the fundamental duel.

A. THE CLASSICAL DUEL

A and B start with loaded weapons and fire their first rounds sumultaneously and then go to the fundamental duel.

$$P(A) = p_A q_B + q_A q_B \; P(A)_f$$

$P(AB) = p_A p_B$ (both may be killed on first round).

Example: Let $X_A \sim ned(r_A)$ and $X_B \sim ned(r_B)$.

$$P(A) = \frac{p_A q_B (p_B r_B + r_A)}{p_A r_A + p_B r_B} .$$

A plot of this follows in which the upper end of each contour terminates at $p_B = 1 - P(A)$ and where A is better off if $q_B r_B > r_A$, $q_B \neq 1$.

B. THE DUEL WITH EQUAL INITIAL SURPRISE (TACTICAL EQUITY)

One-half the time the duel begins with A sighting B first. A then fires one round, which alerts B, and the duel then proceeds as a fundamental duel. The other half of the time, B first first.
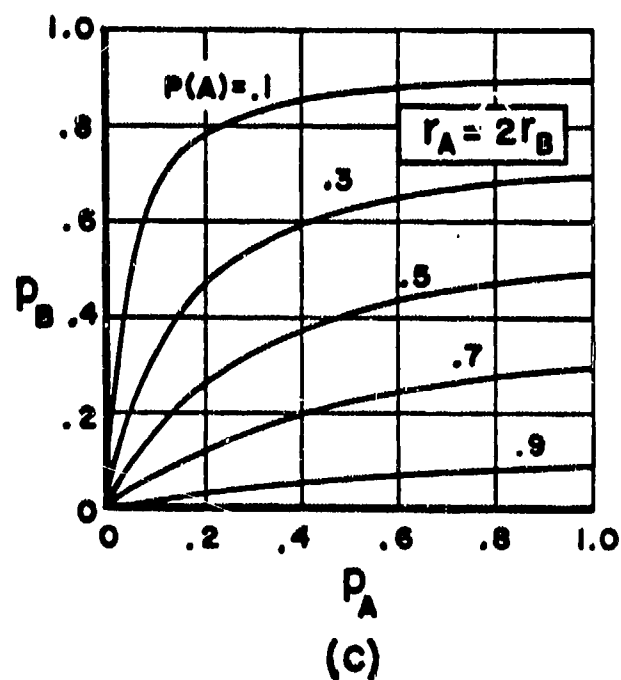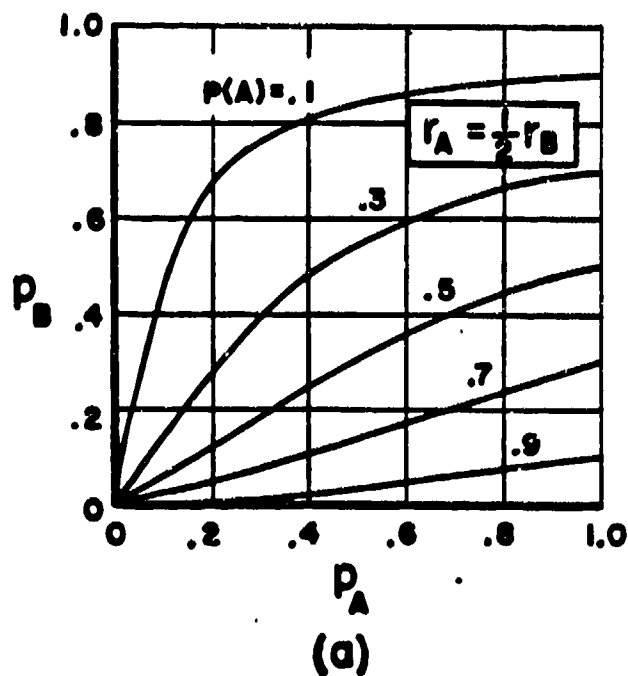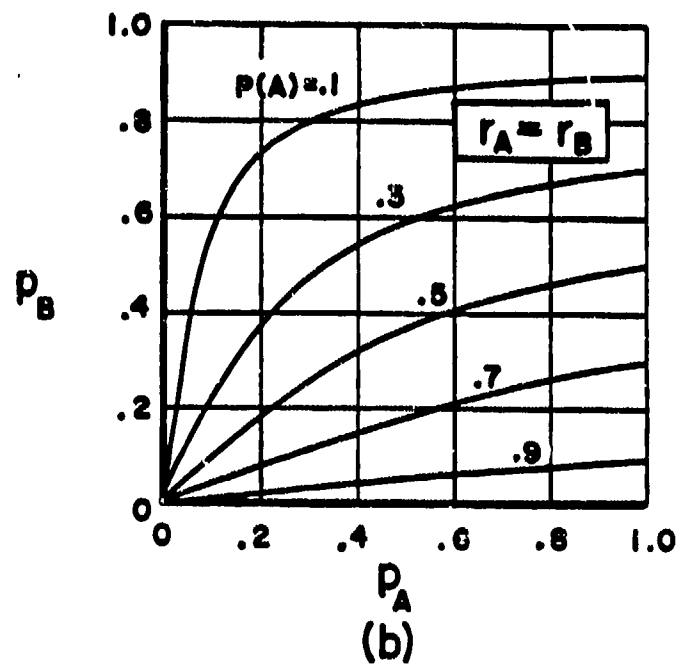
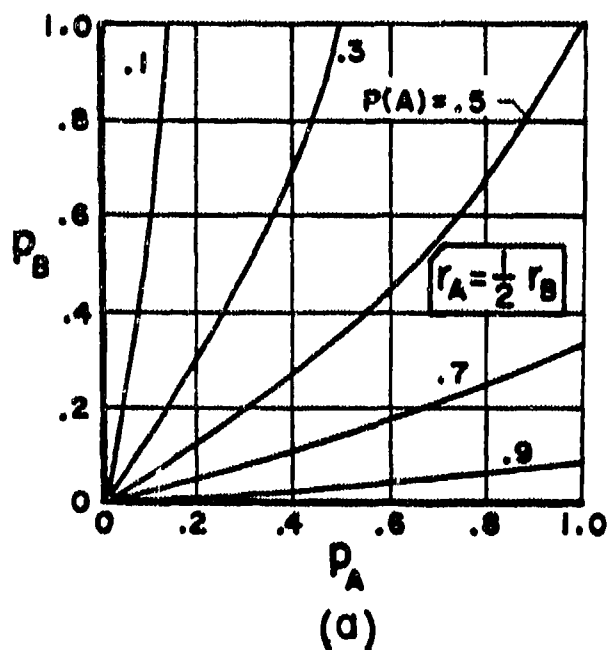The Classical Duel with Negative Exponential
Firing Times

W & A1

$$P(A) = \frac{1}{2}\left[p_A + q_A\, P(A)_f \right] + \frac{1}{2} q_B\, P(A)_f \; .$$
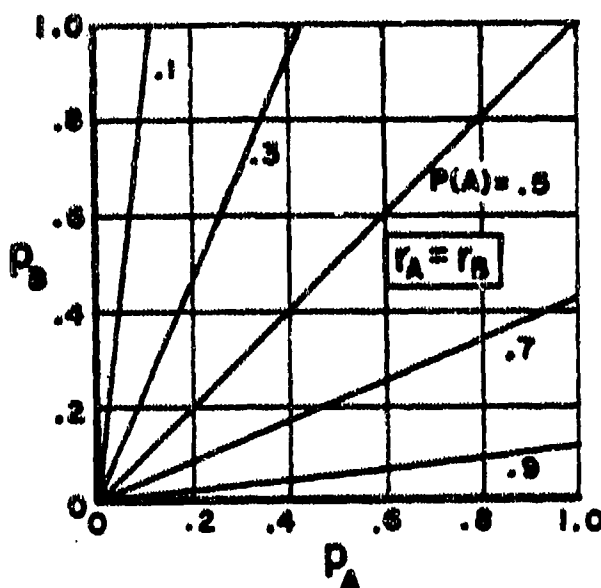
Example 1: Let $X_A \sim ned(r_A)$ and $X_B \sim ned(r_B)$.

$$P(A) = \frac{p_A}{2}\left[\frac{(2 - p_B)r_A + p_B r_B}{p_A r_A + p_B r_B} \right] \; .$$

(a)

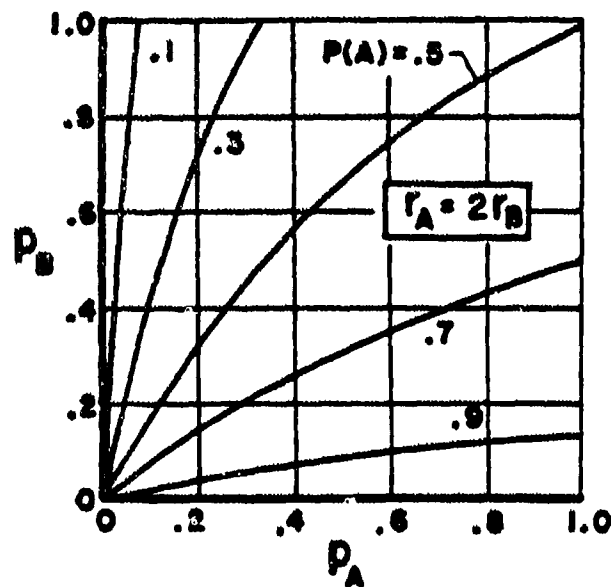The Tactical Equity Duel with Negative
Exponential Firing Times

W & Al

C118

(b)   (c)

The Tactical Equity Duel with Negative Exponential
Firing Times (Continued)   W & A1

Example 2: Let $X_A \sim \text{Erlang}(2, r_A)$ and $X_B \sim \text{Erlang}(2, r_B)$.

$$P(A) = \frac{1}{2} \, p_A \left\{ \frac{(p_A r_A^2 - p_B r_B^2)[2r_A^2 - (r_A^2 + r_B^2)p_B] + 4r_A r_B (r_A + r_B)[2r_A + (r_B - r_A)p_B]}{(p_A r_A^2 - p_B r_B^2)^2 + 4r_A r_B (r_A + r_B)(p_A r_A + p_B r_B)} \right\} \quad A6$$

C. THE DUEL WITH UNEQUAL INITIAL SURPRISE

Let α equal the fraction of the time A sights B first, and,
$1 - \alpha$ be the fraction of the time B initiates the action. Whoever
starts first gets one round without opposition and then a fundamental
duel begins. This is simply a generalization of Section B. above.

$$P(A) = \alpha[p_A + q_A \, P(A)_f] + (1 - \alpha)q_B \, P(A)_f \quad .$$

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D. TACTICAL EQUITY WITH INITIALLY LOADED WEAPONS

Each contestant fires one round first, half of the time.
However, in this case, the opponent has a loaded weapon and immediately returns the fire with one round, thus precipitating the duel
if both survive the opening engagement.

$$P(A) = \frac{1}{2} \{ p_A + q_A q_B \ P(A)_f \} + \frac{1}{2} \{ q_B p_A + q_B q_A \ P(A)_f \}$$

$$= \frac{1}{2} \ p_A (1 + q_B) + q_A q_B \ P(A)_f \ .$$

Example: Let $X_A \sim ned(r_A)$ and $X_B \sim ned(r_B)$.

A6
$$P[A] = \frac{1}{2} \ p_A (1 + q_B) + \frac{q_A q_B \ p_A r_A}{p_A r_A + p_B r_B} \ .$$

E. RANDOM INITIAL SURPRISE

Let

T_S = rv sighting time (not necessarily a positive rv)

$f_S(t)$ = pdf of T_S

$\Theta_S(u)$ = cf of $f_{T_S}(t)$.

The sighting time is a period during which one contestant may fire with
impunity at his opponent. At the end of the sighting-time period, if
the duelist who was firing has not killed his opponent, the fundamental
duel resumes. A positive T_S is a time advantage for A and a

negative T_S is a time advantage for B.

$$P(A) = \frac{1}{2} + \frac{1}{2\pi i} (P) \int_{-\infty}^{\infty} \Phi_A(-u) \, \Phi_B(u) \, \Theta_S(u) \, \frac{du}{u}$$

$$= \frac{1}{2\pi i} \int_L \Phi_A(-u) \, \Phi_B(u) \, \Theta_S(u) \, \frac{du}{u}$$

$$= 1 + \frac{1}{2\pi i} \int_U \Phi_A(-u) \, \Phi_B(u) \, \Theta_S(u) \, \frac{du}{u} \quad .$$

Example 1: Let

$$X_A \sim \text{ned}(r_A), \quad X_B \sim \text{ned}(r_B)$$

$$f_{T_S}(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-t^2/2\sigma^2}$$

$$\Theta_S(u) = e^{-(\sigma^2 u^2)/2} \quad .$$

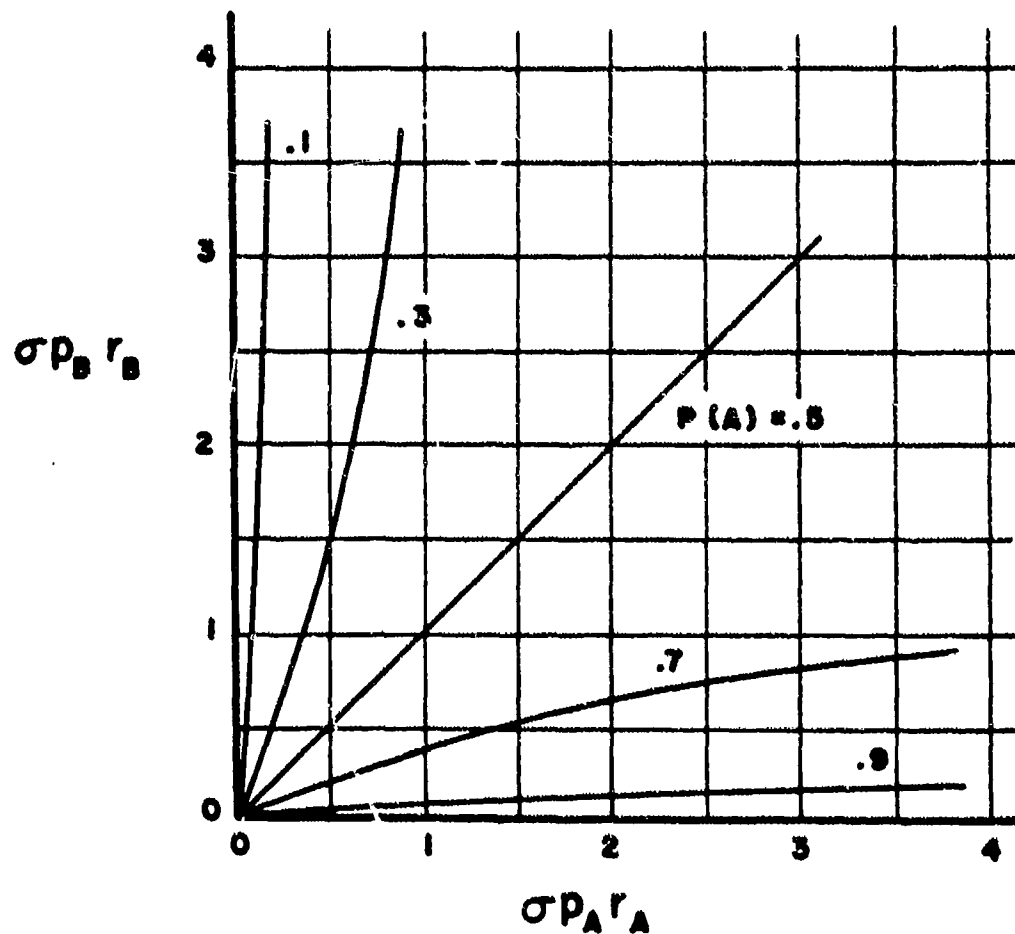$$P(A) = \frac{1}{2} - \frac{\beta}{(\alpha + \beta)} T(\alpha) + \frac{\alpha}{(\alpha + \beta)} T(\beta)$$

where

$$\alpha = \sigma \, p_A \, r_A \qquad \text{and} \qquad \beta = \sigma \, p_B \, r_B$$

$$T(y) = \frac{1}{\sqrt{2\pi}} e^{y^2/2} \int_y^{\infty} e^{-x^2/2} \, dx \quad .$$

This is plotted in the following figure.

$\sigma p_B r_B$

$\sigma p_A r_A$

W & A1

The Duel with Random Initial Surprise
(Negative Exponential Firing Times and Normal Sighting Time)

<u>Example 2</u>: Let

$$X_A \sim \text{ned}(r_A) \quad \text{and} \quad X_B \sim \text{ned}(r_B)$$

$$f_{T_S}(t) = \frac{1}{c} e^{-t/c}, \quad c, t > 0$$

$$f_{T_S}(t) = 0, \qquad t \leq 0 .$$

Thus, A always has the sighting advantage.

$$P(A) = \frac{p_A r_A}{(p_A r_A + p_B r_B)(1 - p_B r_B c)} + \frac{p_A r_A \, p_B r_B}{\left(p_A r_A + \frac{1}{c} \right)\left(p_B r_B - \frac{1}{c} \right)} .$$

Example 3: Let $X_A \sim ned(r_A)$ and $X_B \sim ned(r_B)$.

$$f_{T_S}(t) = \frac{1}{c} e^{-t/c} , \qquad c,t > 0 ,$$

$$\phantom{f_{T_S}(t)} = 0 , \qquad t \leq 0 .$$

Thus, B always has the sighting advantage.

$$P(A) = \frac{p_A r_A}{(p_A r_A + p_B r_B)(1 + p_B r_B c)} .$$

Example 4: Let $X_A \sim ned(r_A)$ and $X_B \sim ned(r_B)$.

$$f_{T_S}(t) = \frac{1}{2c} e^{-|t-d|/c} \qquad \left. \begin{array}{c} - \infty < t < + \infty \\[4pt] c > 0 \\[4pt] - \infty < d < + \infty \end{array} \right\}$$
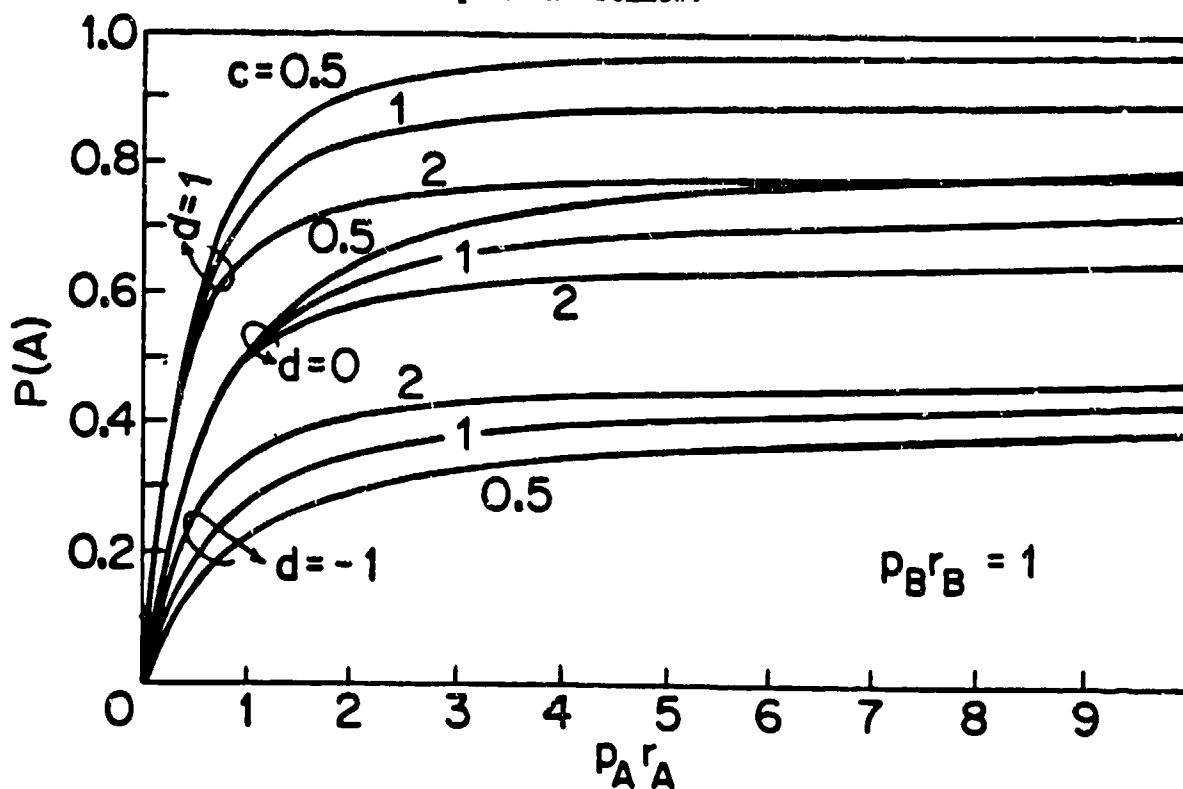
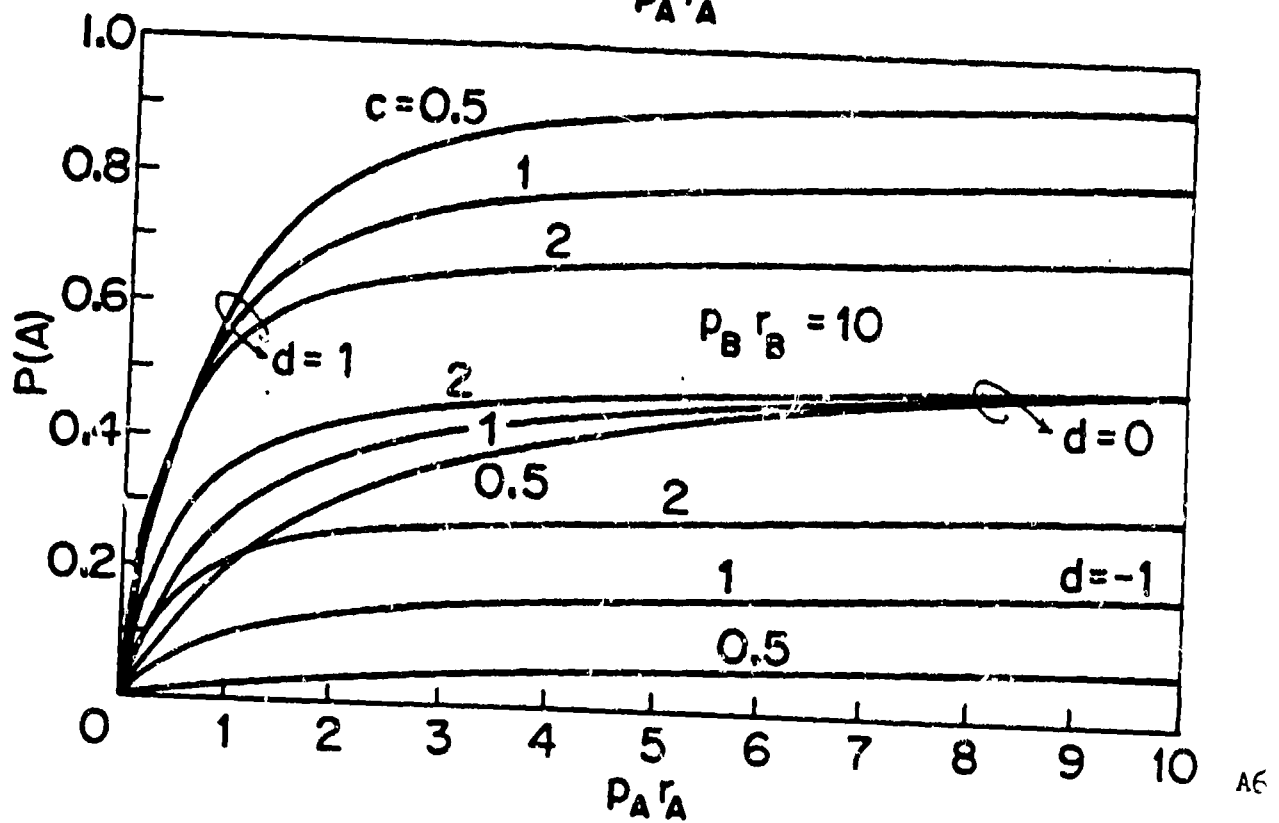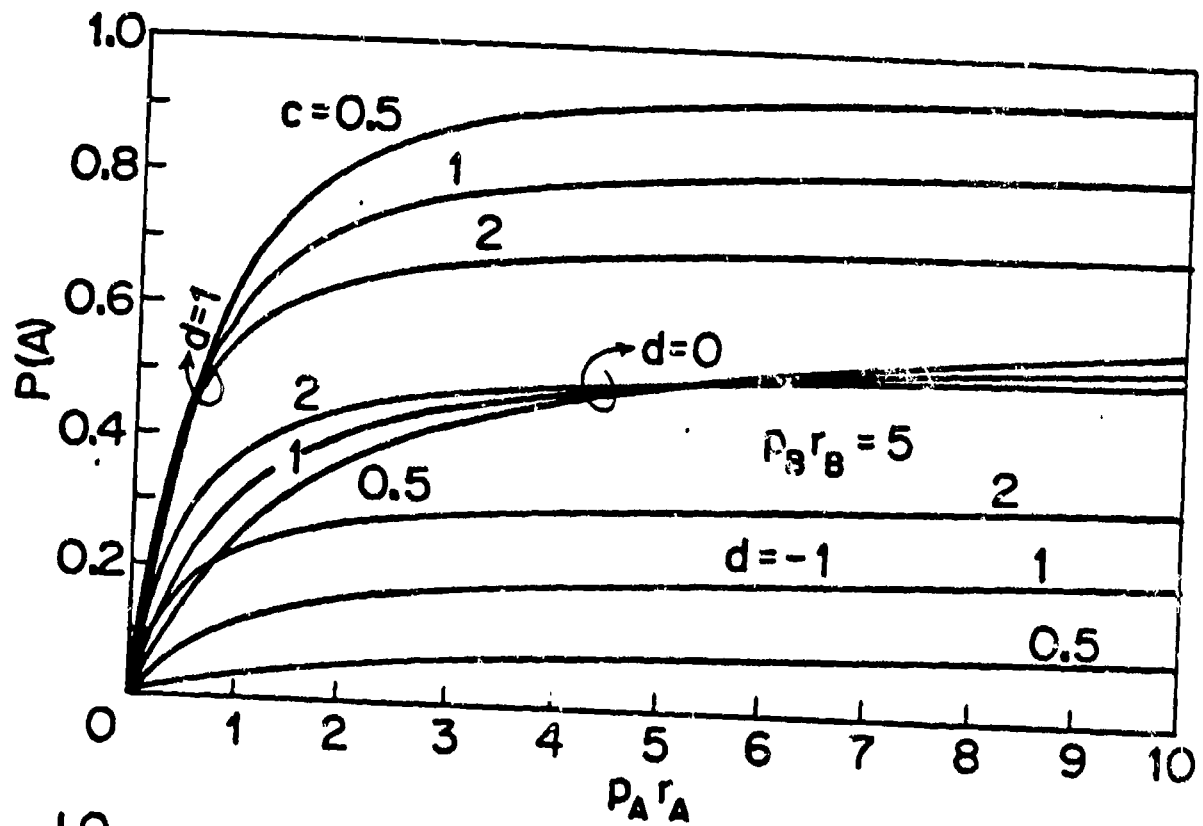$$\Theta_S(u) = \frac{e^{idu}}{1 + c^2 u^2} .$$

$$P(A) = 1 - \frac{p_B r_B \, e^{-p_A r_A d}}{\left(p_B r_B + p_A r_A\right)\left(1 - c^2 \, p_A^2 r_A^2\right)} - \frac{p_A r_A \, p_B r_B \, e^{-d/c}}{2\left(p_B r_B + \frac{1}{c}\right)\left(p_A r_A - \frac{1}{c}\right)}, \quad d \geq 0$$

$$P(A) = \frac{p_A r_A \, e^{p_B r_B d}}{\left(p_A r_A + p_B r_B\right)\left(1 - c^2 \, p_B^2 r_B^2\right)} + \frac{p_A r_A \, p_B r_B \, e^{d/c}}{2\left(p_A r_A + \frac{1}{c}\right)\left(p_B r_B - \frac{1}{c}\right)}, \quad d \leq 0$$

$$P(A) = \frac{p_A r_A}{\left(\frac{1}{c} - p_B r_B\right)}\left[\frac{\frac{2}{c^2}\left(p_A r_A + \frac{1}{c}\right) - p_B r_B\left(p_A r_A + p_B r_B\right)\left(\frac{1}{c} + p_B r_B\right)}{2\left(p_A r_A + p_B r_B\right)\left(\frac{1}{c} + p_B r_B\right)\left(p_A r_A + \frac{1}{c}\right)}\right], \quad d = 0$$

Plots of these equations follow.

III. MULTIPLE HITS TO A KILL

A. FIXED NUMBER OF HITS TO A KILL

A has R_A hits to a kill $(R_A$ fixed$)$

B has R_B hits to a kill $(R_B$ fixed$)$

$$P(A) = \frac{1}{2\pi i} \int_L \Phi_A(-u)\, \Phi_B(u)\, \frac{du}{u}$$

where

$$\Phi_A(u) = \left[\frac{r_A\, \phi_A(u)}{q_A\, \phi_A(u)} \right]^{R_A} .$$

Example: Let $X_A \sim \text{ned}(r_A)$ and $X_B \sim \text{ned}(r_B)$.

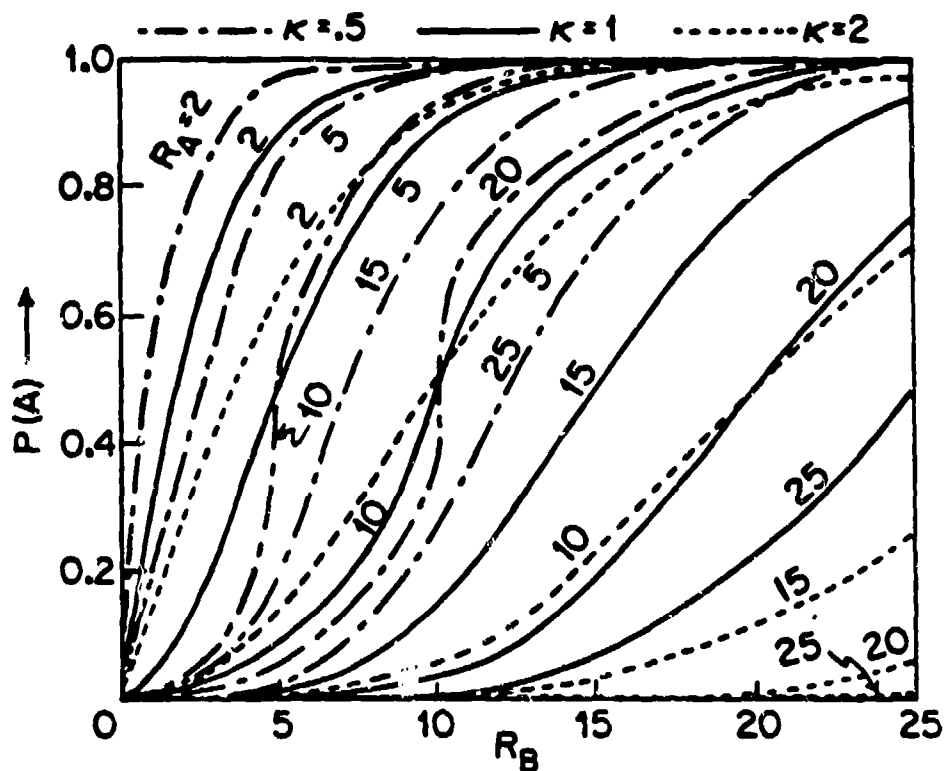$$P(A) = I_\alpha(R_A, R_B)$$

where

$$\alpha = \frac{p_A\, r_A}{p_A r_A + p_B r_B} .$$

Plots of this expression follow.

Let

$$K = \frac{p_B r_B}{p_A r_A}$$

Fundamental Duel - Multiple Hits to a Kill
and Negative Exponential Interfiring Times

Fh7

B. R_A AND R_B ARE RV'S

Let

$$P[R_A = i] = \epsilon_i \qquad \text{and} \qquad P[R_B = j] = \delta_j \ .$$

We have

$$P(A) = \frac{1}{2\pi i} \int_L \phi_A(-u) \, \phi_B(u) \, \frac{du}{u}$$

where

$$\phi_A(u) = \sum_{i=1}^{\infty} \epsilon_i \left(\frac{p_A \, \phi_A(u)}{1 - q_A \, \phi_A(u)} \right)^i \ .$$

Example: Let $X_A \sim \text{ned}(r_A)$ and $X_B \sim \text{ned}(r_B)$
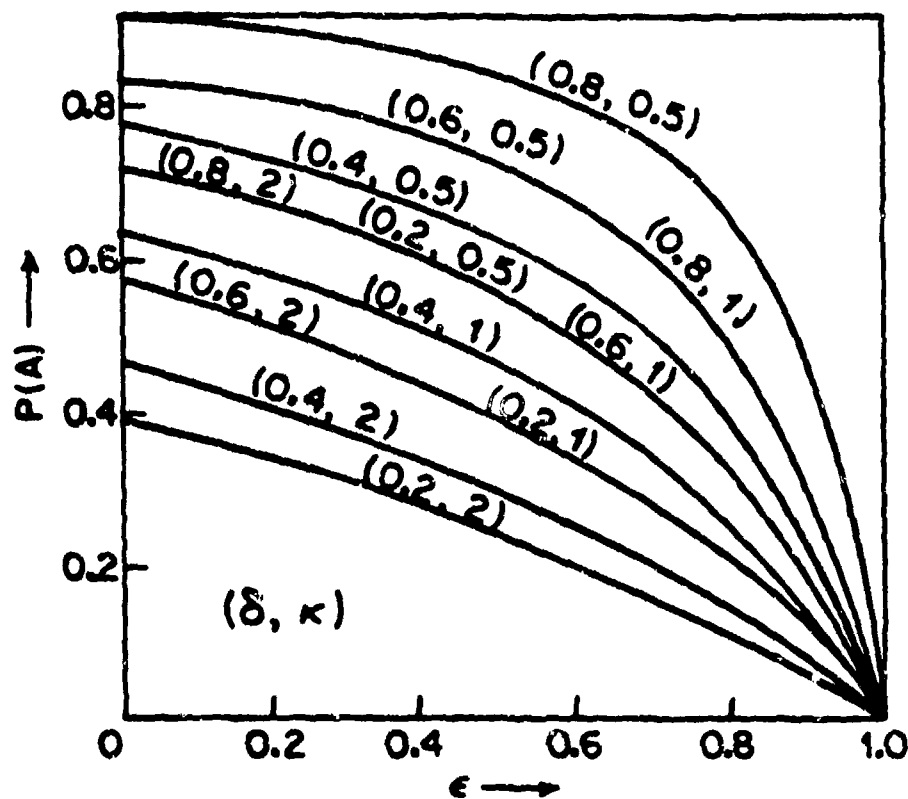
$$\alpha_i = (1 - \epsilon)\epsilon^{i-1}$$

$$\delta_j = (1 - \delta)\delta^{j-1} \ .$$

We have

$$P(A) = \frac{p_A r_A (1 - \epsilon)}{p_A r_A (1 - \epsilon) + p_B r_B (1 - \delta)} \ .$$

Plots of this expression follow in which:

$$K = \frac{p_B r_B}{p_A r_A} \ .$$

C. R_A , R_B FIXED - LIMITED AMMUNITION

Let

$$P[I = i] = \alpha_i \quad \text{and} \quad P[J = j] = \beta_j \ , \quad i, j = 0, 1, 2, \ldots$$

$$P(A) = \frac{1}{2\pi i} \int_L \Phi_{A1}(-u) \, \Phi_{B1}(u) \, \frac{du}{u} + \left(\sum_{j=0}^{R_B - 1} \beta_j + \sum_{j = R_B}^{\infty} \beta_j \sum_{\nu=0}^{R_B - 1} \binom{j}{\nu} p_B^{\nu} q_B^{j-\nu} \right)$$

$$\cdot \left[1 - \left(\sum_{i=0}^{R_A - 1} \alpha_i + \sum_{i = R_A}^{\infty} \alpha_i \sum_{\nu=0}^{R_A - 1} \binom{i}{\nu} p_A^{\nu} q_A^{i-\nu} \right) \right]$$

FD - CRIFT

$$P(AB) = \left(\sum_{i=0}^{R_A-1} \alpha_i + \sum_{i=R_A}^{\infty} \alpha_i \sum_{\nu=0}^{R_A-1} \binom{i}{\nu} p_A^{\nu} q_A^{i-\nu} \right)$$

$$\cdot \left(\sum_{j=0}^{R_B-1} \beta_j + \sum_{j=R_B}^{\infty} \beta_j \sum_{\nu=0}^{R_B-1} \binom{j}{\nu} p_B^{\nu} q_B^{j-\nu} \right)$$

where

$$\phi_{A1}(u) = \left[\frac{p_A \phi_A(u)}{1 - q_A \phi_A(u)} \right]^{R_A} \sum_{i=R_A}^{\infty} \left[1 - I_{q_A \phi_A(u)}(i - R_A + 1, R_A) \right], \quad \text{and}$$

$\phi_{B1}(u)$ is the same formula with A replaced by B.

Example: Let $\alpha_i = (1-\alpha)\alpha^i$, $\beta_j = (1-\beta)\beta^j$, $i, j = 0, 1, \ldots$

$$X_A \sim \text{ned}(r_A) \qquad \text{and} \qquad X_B \sim \text{ned}(r_B)$$

$$P(A) = \left(\frac{\beta \, p_B}{1 - \beta q_B} \right)^{R_B} \sum_{i=C}^{R_B-1} \binom{R_A + i - 1}{i} \left(\frac{r_B(1 - \beta q_B)}{r_A(1 - \alpha q_A) + r_B(1 - \beta q_B)} \right)^{i}$$

$$\cdot \left(\frac{\alpha \, r_A \, p_A}{r_A(1 - \alpha q_A) + r_B(1 - \beta q_B)} \right)^{R_A} + \left[1 - \left(\frac{\beta \, p_B}{1 - \beta q_B} \right)^{R_B} \right] \left(\frac{\alpha \, p_A}{1 - \alpha q_A} \right)^{R_A}$$

$$P(AB) = \left[1 - \left(\frac{\alpha \, p_A}{1 - \alpha q_A} \right)^{R_A} \right] \left[1 - \left(\frac{\beta \, p_B}{1 - \beta q_B} \right)^{R_B} \right] .$$

D. **R_A AND R_B ARE RV'S - LIMITED AMMUNITION**

Let $P[I = i] = \alpha_i$, $P[J = j] = \beta_j$, $i,j = 0,1,\ldots$

$$P[R_A = i] = \epsilon_i, \quad P[R_B = j] = \delta_j, \quad i,j = 0,1,\ldots$$

$$P(A) = \frac{1}{2\pi i} \int_L \phi_{A1}(-u)\, \phi_{B1}(u)\, \frac{du}{u} + P[\bar{K}_B][1 - P[\bar{K}_A]]$$

where

$$\phi_{A1}(u) = \sum_{j=1}^{\infty} \epsilon_j \left[\frac{p_A \phi_A(u)}{1 - q_A \phi_A(u)} \right]^j \sum_{i=j}^{\infty} \alpha_i \left(1 - I_{q_A \phi_A(u)}(i - j + 1, j) \right)$$

$$P[\bar{K}_A] = \sum_{j=1}^{\infty} \epsilon_j \left[\sum_{i=0}^{j-1} \alpha_i + \sum_{i=j}^{\infty} \alpha_i \sum_{\nu=0}^{j-1} \binom{i}{\nu} p_A^\nu q_A^{i-\nu} \right]$$

$$P(AB) = P[\bar{K}_A]\, P[\bar{K}_B] \quad .$$

Example: Let $\alpha_i = (1-\alpha)\alpha^i$, $\beta_j = (1-\beta)\beta^j$, $i,j = 0,1,2,\ldots$

$$\epsilon_i = (1-\epsilon)\epsilon^i, \quad \delta_j = (1-\delta)\delta^j, \quad i,j = 0,1,2,\ldots$$

$$X_A \sim \text{ned}(r_A) \qquad \text{and} \qquad X_B \sim \text{ned}(r_B)$$

$$P(A) = \left[\frac{\alpha \epsilon\, p_A r_A}{r_A[1 - \alpha(1-p_A \epsilon)] + r_B[1 - \beta(1-p_B \delta)]} \right] \left[\frac{\beta\, p_B\, \delta}{1 - \beta(1 - p_B\, \delta)} \right]$$

$$+ \left[\frac{1-\beta}{1 - \beta(1 - p_B\, \delta)} \right] \left[\frac{\alpha\, p_A\, \epsilon}{1 - \alpha(1 - p_A\, \epsilon)} \right]$$

FD - CRIFT

Kw & Bl
$$P(AB) = \left[\frac{1 - \alpha}{1 - \alpha(1 - p_A \varepsilon)}\right]\left[\frac{1 - \beta}{1 - \beta(1 - p_B \delta)}\right]$$

E. DAMAGE

1. Damage as a Function of Round Number

See FM - CRIFT (page C26 et seq.) for definitions.

Example: (same as in FM - CRIFT, page C28)

$$X_A = \text{ned}(r_A) \quad\text{and}\quad X_B = \text{ned}(r_B),$$

$$p_{D_A}(0) = \alpha_A, \quad p_{D_A}(1) = \beta_A, \quad p_{D_A}(b) = \gamma_A, \quad \alpha_A + \beta_A + \gamma_A = 1$$

$$p_{D_B}(0) = \alpha_B, \quad p_{D_B}(1) = \beta_B, \quad p_{D_B}(a) = \gamma_B, \quad \alpha_B + \beta_B + \gamma_B = 1$$

where a, b are the maximum tolerable damages for A and B, respectively.

$$P(A) = \frac{r_A \gamma_A}{r_A \gamma_A + r_B \gamma_B} + \frac{r_A^b \beta_A^b}{(r_A \gamma_A + r_B \gamma_B)} \cdot \frac{r_B \gamma_B}{[r_A(1 - \alpha_A) + r_B \gamma_B]^b}$$

$$- \frac{r_A \gamma_A}{(r_A \gamma_A + r_B \gamma_B)} \cdot \frac{r_B^a \beta_B^a}{[(1 - \alpha_B)r_B + r_A \gamma_A]^a}$$

$$+ (r_A \beta_A)^b (r_B \beta_B)^a$$

$$\cdot \left[\frac{r_A \gamma_A}{(r_A \gamma_A + r_B \gamma_B)(r_A \beta_A)^b [r_A \gamma_A + r_B(1 - \alpha_B)]^a} + C\right]$$

C132

where

$$C = \sum_{i+j \leq b-2} \binom{b-i-j+a-3}{a-1}$$

$$\cdot \frac{1}{(r_A \beta_A)^i [r_B \gamma_B + (1-\alpha_A)r_A]^j [(1-\alpha_B)r_B + (1-\alpha_A)r_A]^{b-i-j-1}}$$

$$- \sum_{i+j \leq b-1} \binom{b-i-j+a-2}{a-2}$$

$$\cdot \frac{1}{(r_A \beta_A)^i [r_B \gamma_B + (1-\alpha_A)r_A]^j [(1-\alpha_B)r_B + (1-\alpha_A)r_A]^{b-i-j}} \quad .$$

2. **Damage is Time-Homogeneous**

Refer to FM - CRIFT, page C28, for definitions, and a, b are maximum tolerable damages to A and B, respectively.

$P(A) = $ coeff of series (in z and w) expansion of $z^{b-1} w^{a-1}$

$$\frac{p_{D_A} - z G_{D_A}(z)}{(1-z)(1-w)[p_{D_A} + p_{D_B} - z G_{D_A}(z) - w G_{D_B}(w)]} \quad .$$

Example: Let $p_{D_A}(1) = \alpha_A$, $p_{D_A}(b) = \beta_A$, $\alpha_A + \beta_A = p_A$,

$$p_{D_B}(1) = \alpha_B, \quad p_{D_B}(\text{a}) = \beta_B, \quad \alpha_B + \beta_B = p_B,$$

and let

$$x = \frac{\alpha_A}{p_A + p_B} \qquad \text{and} \qquad y = \frac{\alpha_B}{p_A + p_B}.$$

N & J1
$$P(A) = \frac{p_A}{p_A + p_B} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} \binom{i+j}{i} x^i y^j - \sum_{i=0}^{b-2} \sum_{j=0}^{a-1} \binom{i+j}{i} x^{i+1} y^j .$$

IV. ROUND-DEPENDENT HIT PROBABILITIES

A. UNLIMITED AMMUNITION

Let

$$p_n = P(H \text{ on } n\text{-th round} \mid n\text{-th round fired})$$

$$p_N(n) = \rho_n = P(H \text{ on } n\text{-th round}, n\text{-th round fired})$$

$$\rho_n = p_n \prod_{j=0}^{n-1} (1 - p_j) = p_n \prod_{j=0}^{n-1} q_j, \quad \text{where } q_j = 1 - p_j, q_0 = 1$$

$$\Phi_k(u) = \sum_{n=1}^{\infty} \rho_{kn} \phi_k^n(u) \quad \begin{cases} \rho_{kn} = \rho_n & \text{for } k \\ q_{kj} = q_j & \text{for } k \end{cases}, \quad k = A, B$$

$$= \phi_k(u) - [1 - \phi_k(u)] \sum_{i=1}^{n} \phi_k^n(u) \prod_{j=1}^{n} q_{jk} ,$$

Bh5
& A4
$$P(A) = \frac{1}{2\pi i} \int_L \Phi_A(-u) \Phi_B(u) \frac{du}{u} .$$

Example 1: Let $X_A \sim \text{ned}(r_A)$ for A, then:

$$q_{Aj} = \left(\frac{N}{j} - 1\right) a = \frac{(N-j)}{j} a; \quad j = 1,2,\ldots,N,$$

$$= 0 \quad ; \quad j = N+1, N+2, \ldots,$$

where

$$a \leq \frac{1}{N-1}; \quad N \in I^+, \quad a > 0, \quad N, a \text{ constants}.$$

$$q_{A1} = q_A = a(N-1).$$

Let $X_B \sim \text{ned}(r_B)$ for B, p_B a constant.

$$P(A) = 1 - \frac{1}{1+x}\left[1 + \left(\frac{q_A}{N-1}\right)\left(\frac{x}{x+1}\right)\right]^{N-1}$$

where

$$x = \frac{r_A}{p_B r_B}.$$

Example 2: Let $X_A \sim \text{Erlang}(2, r_A)$ for A

$$q_{Aj} = \frac{q_A}{j}; \quad j = 1,2,\ldots.$$

Let $X_B \sim \text{ned}(r_B)$ for B, p_B a constant.

$$P(A) = \frac{(1 + x)^2 - (1 + 2x)\exp\left[q_A\left(\frac{x}{1 + x}\right)^2\right]}{(1 + x)^2}$$

where

$$x = \frac{r_A}{p_B r_B} \ .$$

Example 3: Let $X_A \sim ned(r_A)$ for A, then

$$q_{Aj} = \frac{q_A}{j} \ ; \qquad j = 1, 2, \ldots \ .$$

Let $X_B \sim ned(r_B)$ for B, p_B a constant.

$$P(A) = \frac{x + 1 - \exp\left[q_A\left(\frac{x}{x + 1}\right)\right]}{1 + x}$$
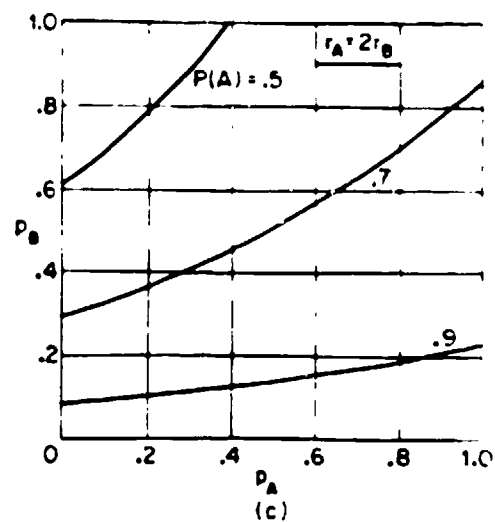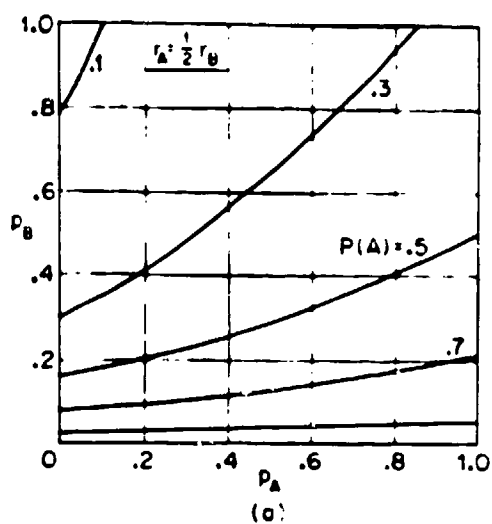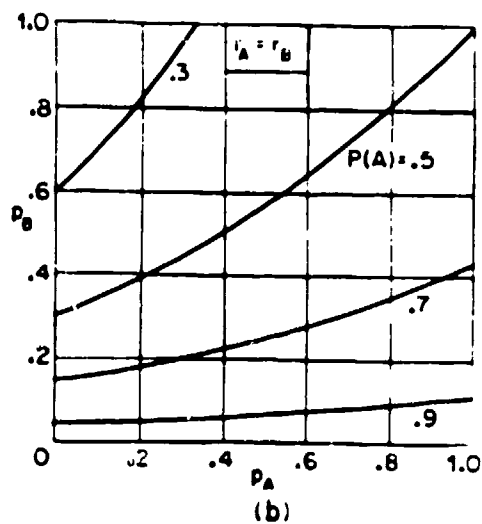
where

$$x = \frac{r_A}{p_B r_B} \ .$$

See the figures for plots of $P(A)$ which follow. $P(A)$ has a lower bound $\neq 0$. The bound is:

$$P(A)_{min} = 1 - \left(\frac{r_B}{r_A + r_B}\right) e^{\frac{r_A}{r_A + r_B}}$$

which, for (a), (b), and (c) in the figures, are .0695, .1756, and .351, respectively.

Plot for Example 3

AL

Example 3 (cont'd): Compare this to FD. Let $P(A)_f$ = the solution to FD with all ned's and $P(A)$ as above. Then

$$\frac{P(A)}{P(A)_f} = \frac{\left(1 + \frac{1}{P_A x}\right)\left\{x + 1 - \exp\left[\frac{x(1 - P_A)}{1 + x}\right]\right\}}{1 + x}$$

where

$$x = \frac{r_A}{P_B r_B} \quad .$$

A4   The above is plotted on the following page.

Example 4: Let $X_A \sim \text{ned}(r_A)$ and $X_B \sim \text{ned}(r_B)$

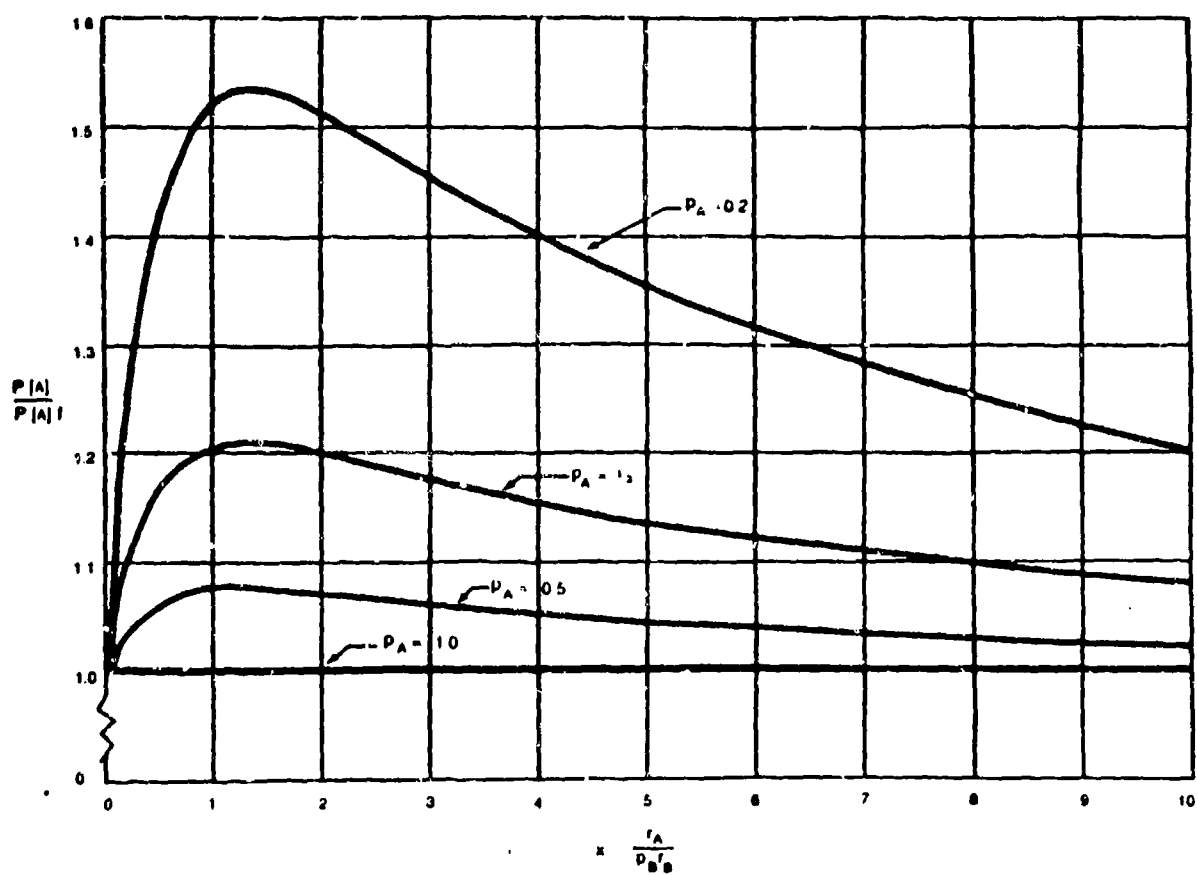$$P_{N_A}(n_A) = \rho_{An} * \binom{n_A + k_A - 1}{k_A} \xi_A^{k_A+1}\left(1 - \xi_A\right)^{n_A-1} \quad \text{for}$$

$$n_A = 1,2,\dots; \quad k_A \le 0; \quad 0 < \xi_A < 1$$

$$P_A \overset{\Delta}{=} \frac{\xi_A}{1 + k_A(1 - \xi_A)} = \frac{1}{\mu_{N_A}}; \quad \text{n.b.} \quad P_A \ne P_1 \quad .$$

Similarly for B.

     Let

W2 $$P_O = \frac{\xi_B r_B}{\xi_A r_A + \xi_B r_B} \quad \text{and} \quad Q_O = 1 - P_O \quad ,$$

FD - CRIFT

$$P(A) = \sum_{i=0}^{k_A} \binom{k_A}{i} \xi_A^{k_A - i} \left(1 - \xi_A\right)^i$$

$$\cdot \sum_{j=0}^{k_B} \binom{k_B}{j} \xi_B^{k_B - j} \left(1 - \xi_B\right)^j I_{Q_0}(i+1, j+1)$$

where

$$I_{Q_0}(i+1, j+1) = P[\text{Bin}(i+j+1, P_0) \le j] .$$

Thus, I_{Q_0} may be found either in binomial probability tables or Incomplete Beta Function Tables.

Special Cases:

(1) $k_B = 0$ or $p_B = 1$; $k_A = \infty \implies \xi_A = 1$

$$P(A) = Q_0 \, e^{-(q_A/p_A)P_0} .$$

(2) $k_A = k_B = \infty \implies \xi_A = \xi_B = 1$

$$P(A) = e^{-\{(q_A/p_A)P_0 - (q_B/p_B)Q_0\}}$$

$$\cdot \left\{ Q_0 \sum_{i=0}^{\infty} \frac{\left(\frac{q_A}{p_A} P_0 \frac{q_B}{p_B} Q_0\right)^i}{i!} + \sum_{i=1}^{\infty} \frac{\left(\frac{q_B}{p_B} Q_0\right)^i}{i!} \sum_{j=0}^{i-1} \frac{\left(\frac{q_A}{p_A} P_0\right)^j}{j!} \right\} .$$

(3) $k_A = \infty; \quad k_B = 1 \quad \Rightarrow \quad \xi_A = 1$

$$P(A) = Q_0 \, e^{-(q_A/p_A)P_0} \left\{ 1 + P_0 \left(1 - \xi_B \right) \left(1 + \frac{q_A}{p_A} Q_0 \right) \right\} \ .$$

On the following pages are the plots of $P(A)$.

B. LIMITED AMMUNITION

1. General IFT's

a. Fixed Ammunition Supply k for A and ℓ for B

$$\Phi_{A1}(u) = \sum_{n=1}^{k} \rho_{An} \, \phi_A^n(u) = \sum_{n=1}^{k} p_{An} \prod_{j=0}^{n-1} q_{Aj} \, \phi_A^n(u)$$

$$P(A) = \frac{1}{2\pi i} \int_L \Phi_{A1}(-u) \, \Phi_{B1}(u) \, \frac{du}{u} + \prod_{j=1}^{\ell} q_{Bj} \left[1 - \prod_{i=0}^{k} q_{Ai} \right]$$

$$P(AB) = \prod_{i=1}^{k} q_{Ai} \prod_{j=1}^{\ell} q_{Bj} \ .$$

Example 1: Let $X_A \sim \text{ned}(r_A)$ and $X_B \sim \text{ned}(r_B)$.

$$P(A) = \sum_{i=1}^{k} p_{Ai} \prod_{\nu=0}^{i-1} q_{A\nu} \sum_{j=1}^{\ell} p_{Bj} \prod_{\xi=0}^{j-1} q_{B\xi} \, I_{\frac{r_A}{r_A + r_B}}(i,j)$$

$$+ \prod_{j=0}^{\ell} q_{Bj} \left[1 - \prod_{i=0}^{k} q_{Ai} \right] \ .$$

(b)

(a)

(c)

$k_{A:0}$ $k_{B:1}$

$r_A = r_B$

P(A) = .5

(b)

$r_A = 1/2 r_B$

P(A) = .5

(a)

$r_A = 2 r_B$

P(A) = .5

(c)

$k_A = 0 \qquad k_B = \infty$

(b)

(a)

(c)

(b)

(a)

(c)

$k_{A = 1} \quad k_{B = 1}$

$r_A = r_B$

(b)

$r_A = 1/2 \, r_B$

(a)

$r_A = 2 \, r_B$

(c)

(b)

(a)

(c)

$k_A = \infty \quad k_B = 0$

$r_A = r_B$

$P(A) = .5$

p_B

p_A

(b)

$r_A = 1/2\, r_B$

$P(A) = .5$

p_B

p_A

(a)

$r_A = 2\, r_B$

$-P(A) = .5$

p_B

p_A

(c)

$k_A = \infty \qquad k_B = 1$

$r_A = r_B$

(b)

$r_A = 1/2\, r_B$

(a)

$r_A = 2\, r_B$

(c)

$k_A = \infty$ $k_B = \infty$

$r_A = r_B$

(b)

$r_A = 1/2 \, r_B$

(a)

$r_A = 2 \, r_B$

(c)

W2
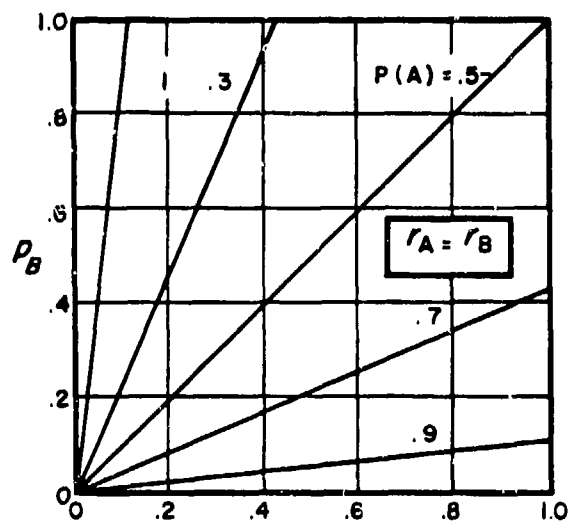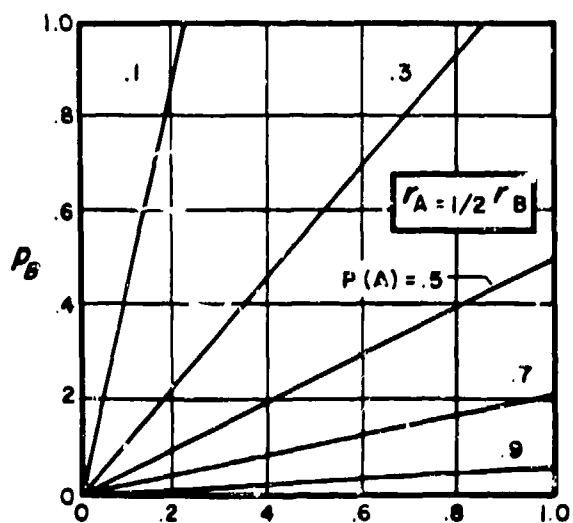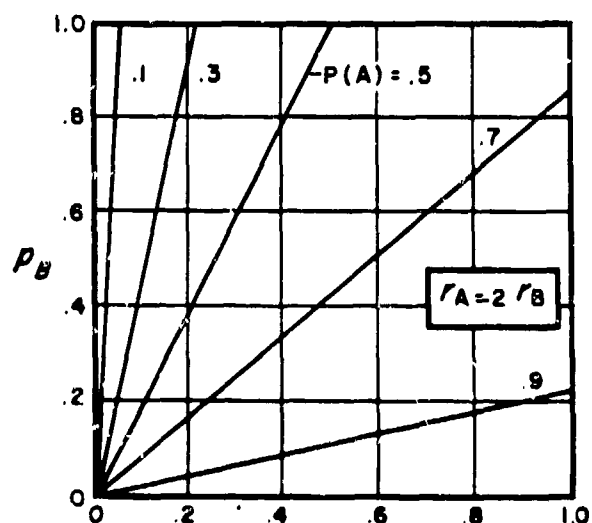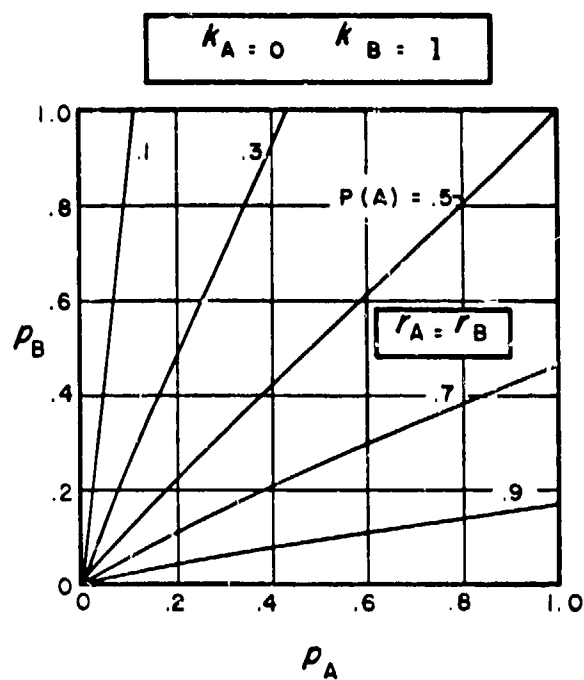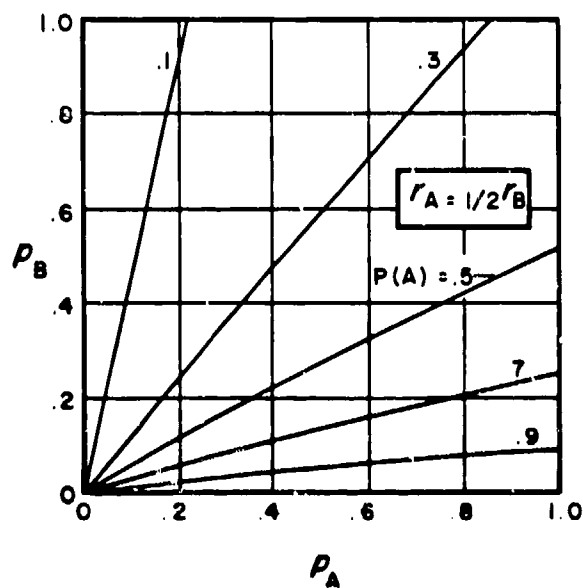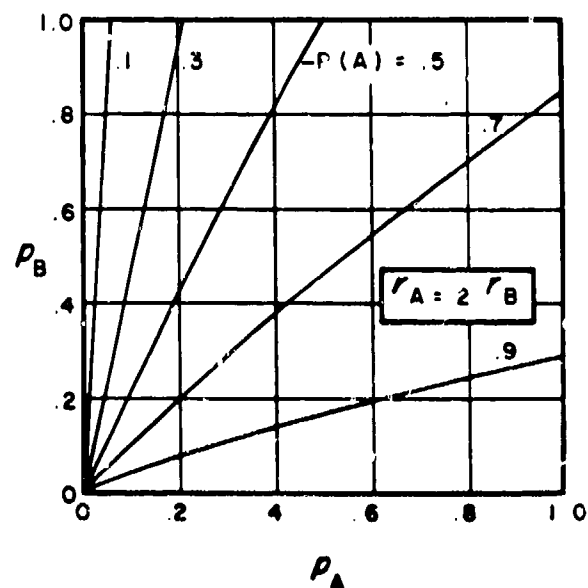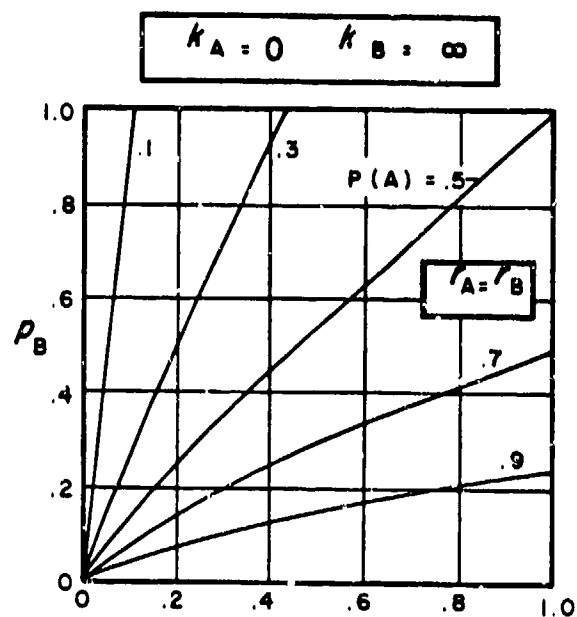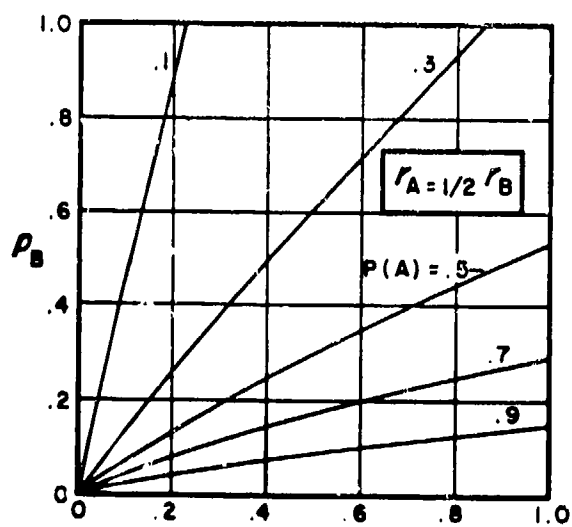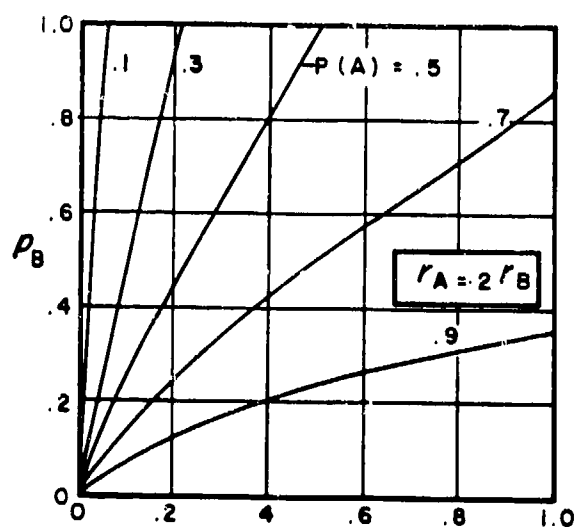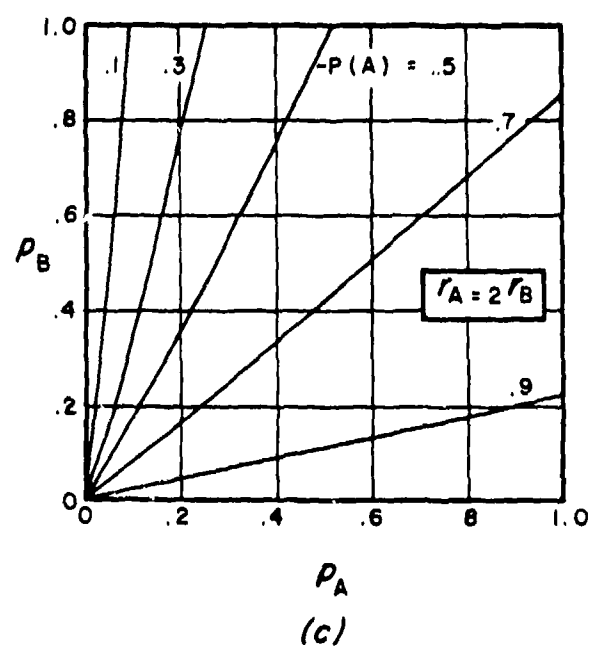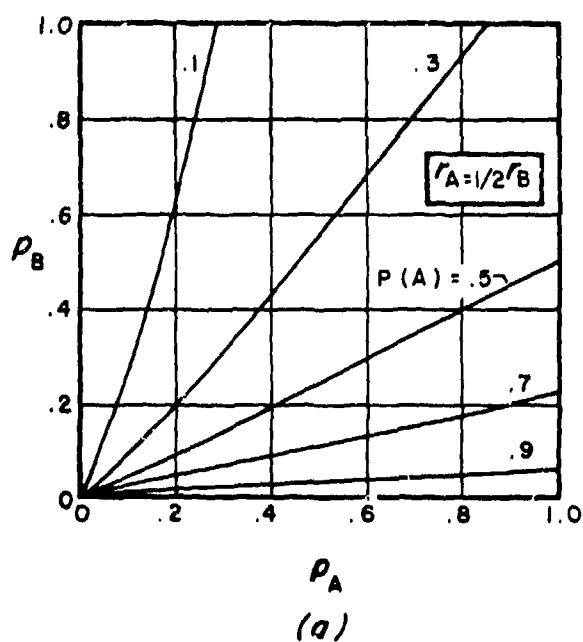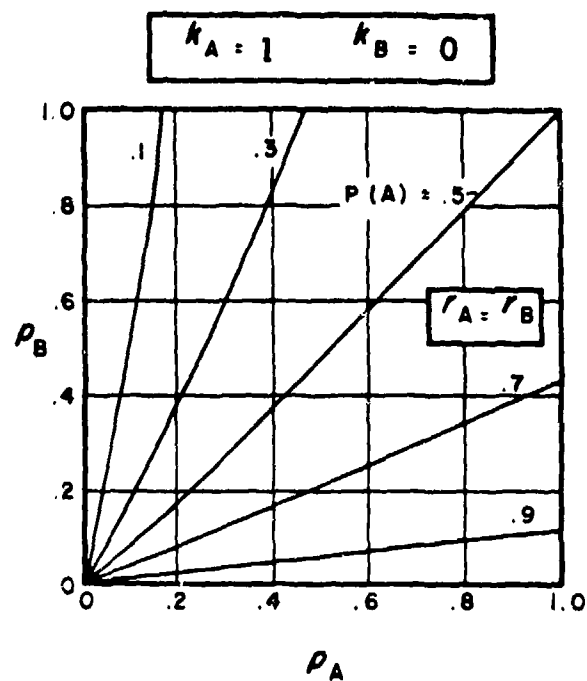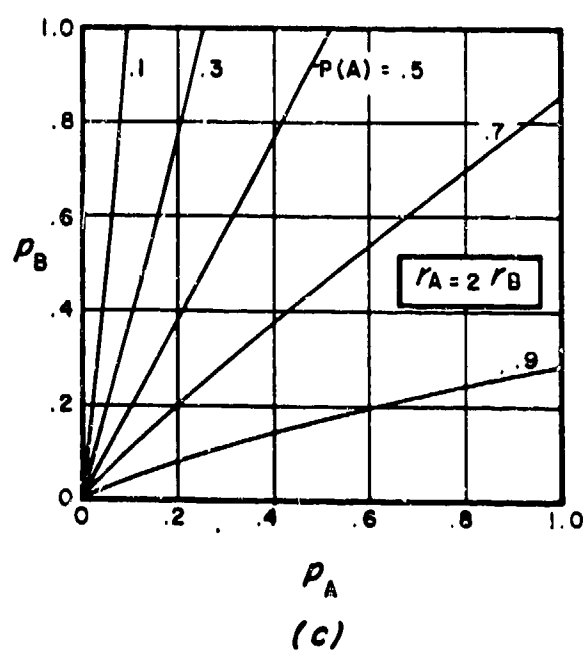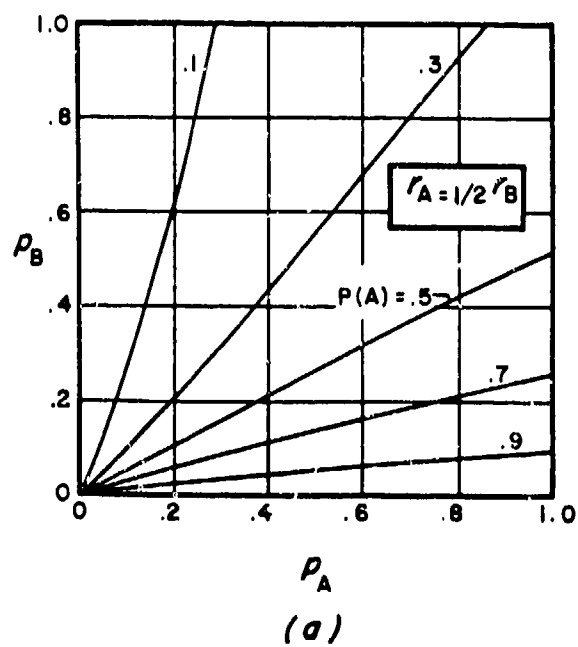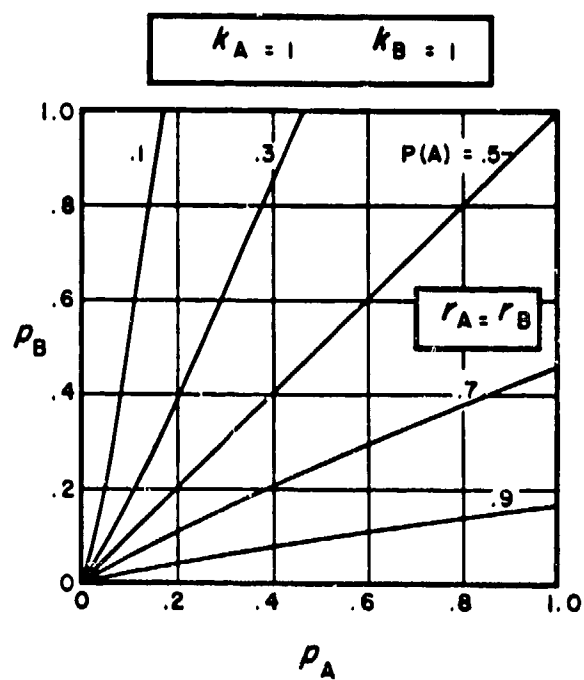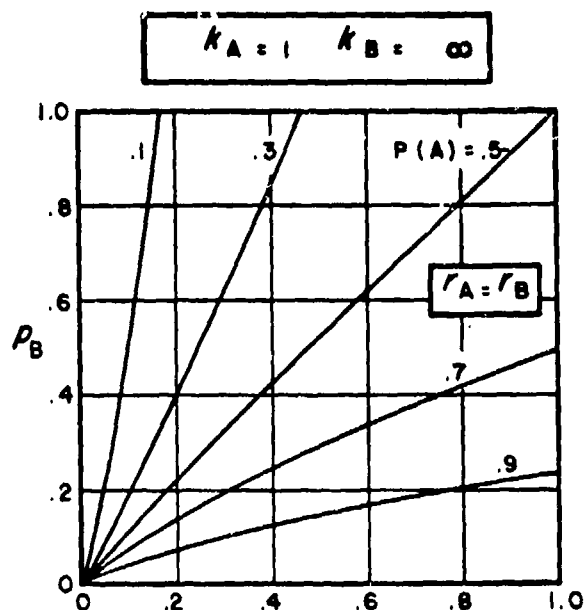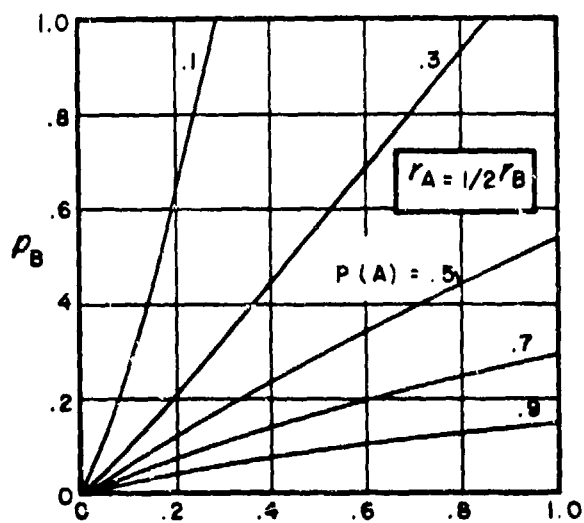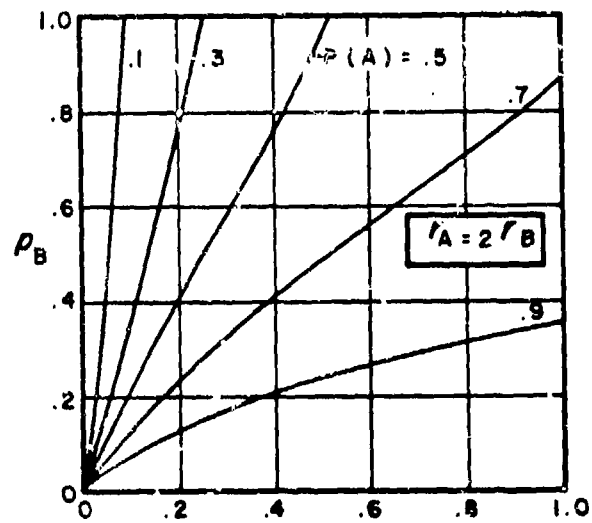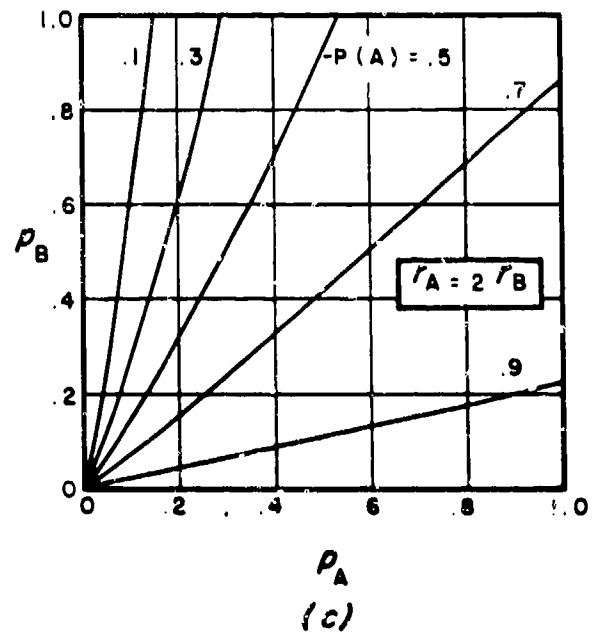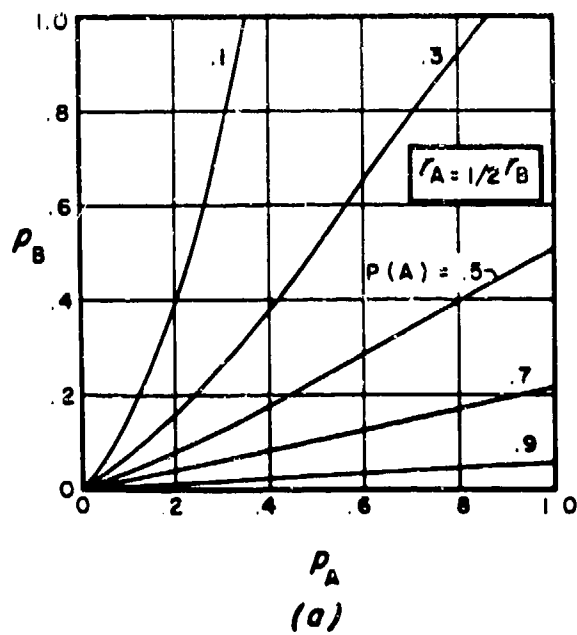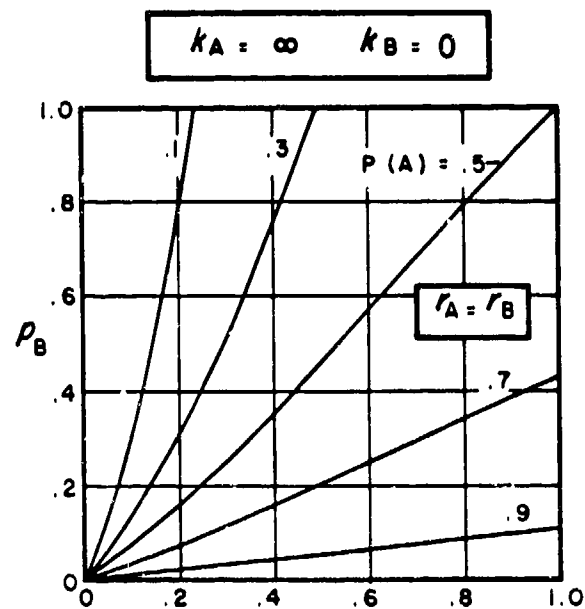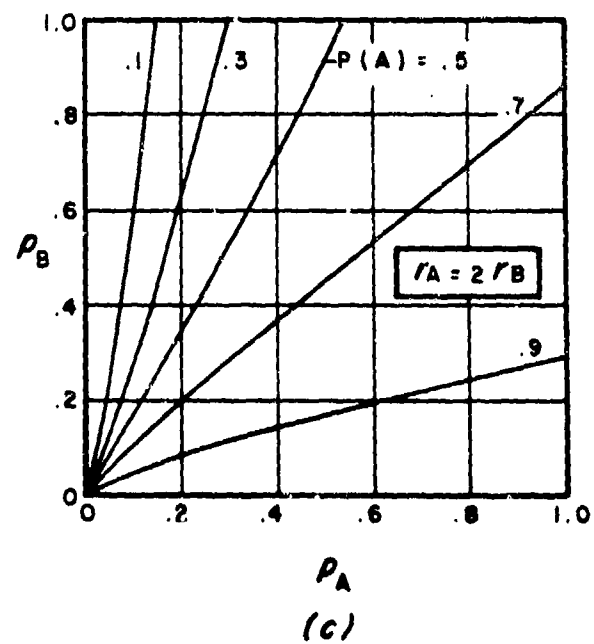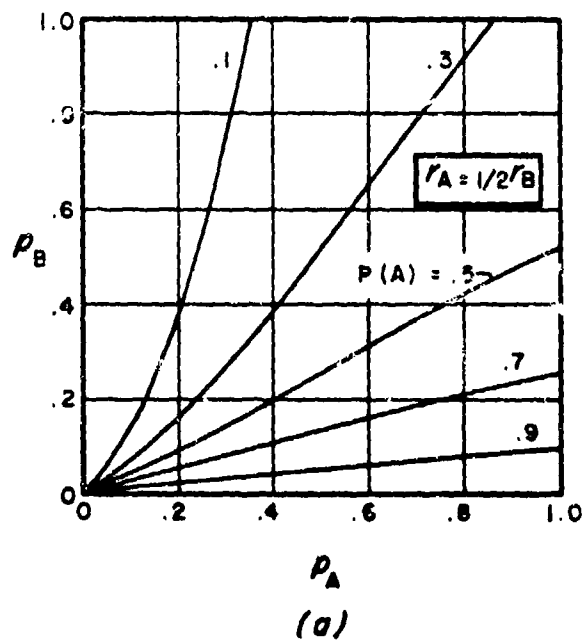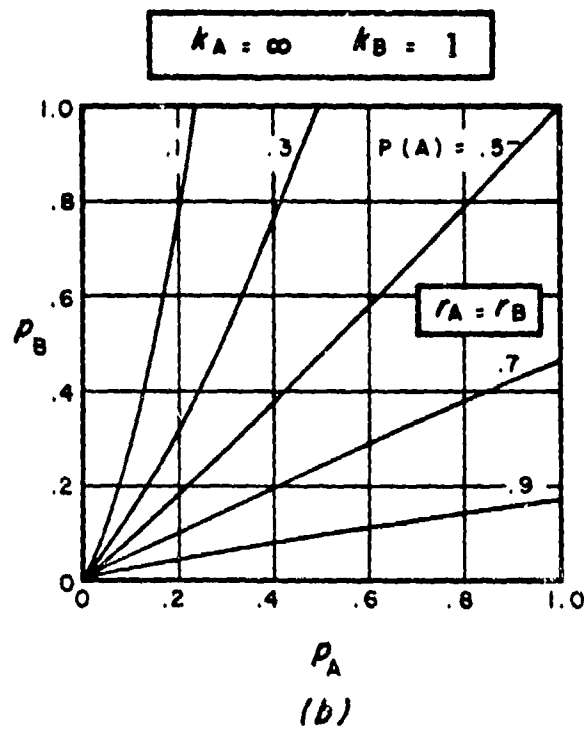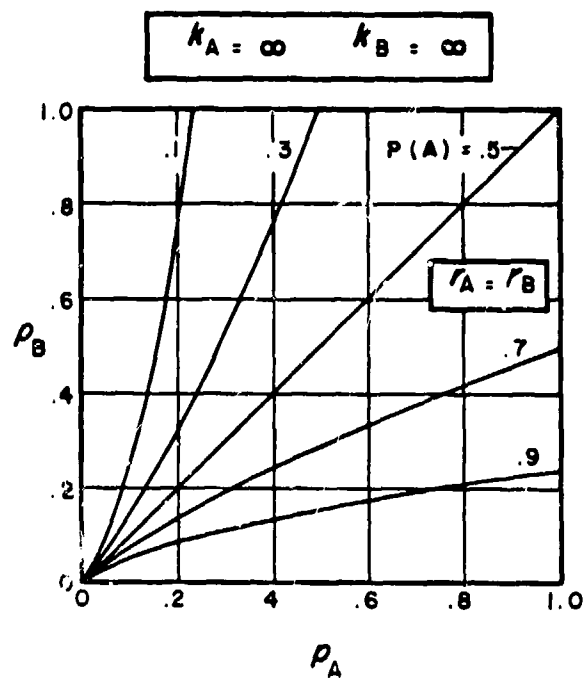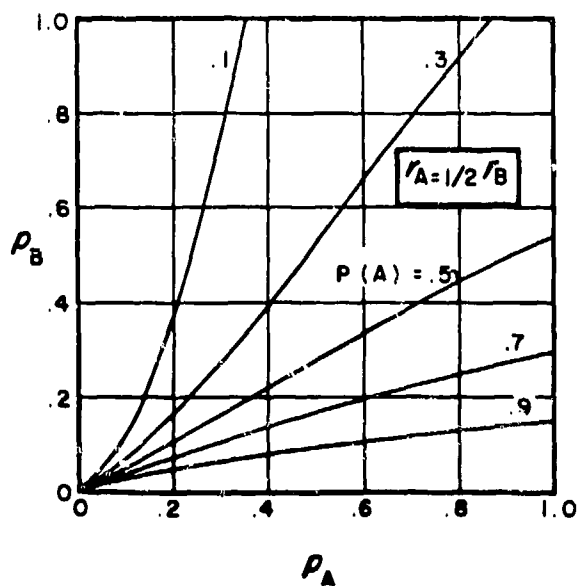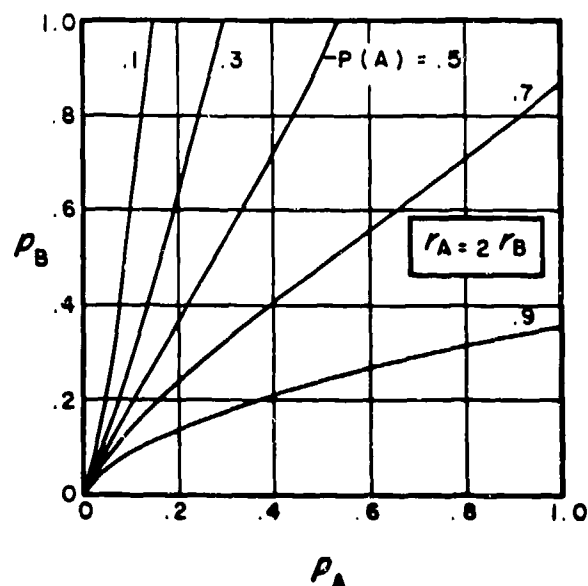
Example 2: Let $X_A \sim \text{Erlang}(m ; r_A)$ and $X_B \sim \text{Erlang}(n ; r_B)$.

$$P(A) = \sum_{i=1}^{k} p_{Ai} \prod_{\nu=0}^{i-1} q_{A\nu} \sum_{j=1}^{\ell} p_{Bj} \prod_{\xi=0}^{j-1} q_{B\xi} I_{\frac{mr_A}{mr_A + nr_B}}(mi, nj) \qquad \text{Bh5}$$

$$+ \prod_{j=0}^{\ell} q_{Bj} \left[1 - \prod_{i=0}^{k} q_{Ai} \right] .$$

b. Ammunition Supply a RV

$$P[I = i] = \alpha_i , \quad \sum_{i=0}^{\infty} \alpha_i = 1 \qquad \text{and} \qquad P[J = j] = \beta_j , \quad \sum_{j=0}^{\infty} \beta_j = 1$$

$$\phi_{A1}(u) = \sum_{i=1}^{\infty} \alpha_i \sum_{n=1}^{i} \rho_{Ai} \phi_A^i(u)$$

$$P(A) = \frac{1}{2\pi i} \int_L \phi_{A1}(-u) \phi_{B1}(u) \frac{du}{u}$$

$$+ \sum_{j=0}^{\infty} \beta_j \prod_{\xi=0}^{j} q_{B\xi} \left[1 - \sum_{i=0}^{\infty} \alpha_i \prod_{\nu=0}^{i} q_{A\nu} \right]$$

$$P(AB) = \left(\sum_{i=0}^{\infty} \alpha_i \prod_{\nu=0}^{i} q_{A\nu} \right) \left(\sum_{j=0}^{\infty} \beta_j \prod_{\xi=0}^{j} q_{B\xi} \right) . \qquad \text{Bh5}$$

2. ned IFT's

a. Both Have Ammunition Limitation; A Has k Rounds, B Has ℓ Rounds

Let $X_A \sim \text{ned}(r_A)$ and $X_B \sim \text{ned}(r_B)$

$$C = r_A + r_B - iu$$

$$C' = r_A(1 - \delta_{i,\ell}) + r_B(1 - \delta_{j,k}) - iu \quad .$$

$$\psi_0(u) = \sum_{i=0}^{n} \sum_{j=0}^{m} \prod_{\nu=0}^{i} q_{A\nu} \prod_{\xi=0}^{j} q_{B\xi} \frac{1}{C'} \left(\frac{r_A}{C}\right)^{i} \left(\frac{r_B}{C}\right)^{j} \binom{i+j-1}{j}$$

$$+ \sum_{j=1}^{m} \prod_{i=0}^{k} q_{A1} \prod_{\xi=0}^{j} q_{B\xi} \left(\frac{1}{r_B - iu}\right) \left[\frac{r_B}{r_B - iu}\right]^{j} I_{\frac{r_A}{C}}(k, j+1)$$

$$+ \sum_{i=1}^{n} \prod_{j=0}^{\ell} q_{Bj} \prod_{\nu=0}^{i} q_{A\nu} \left(\frac{1}{r_A - iu}\right) \left[\frac{r_A}{r_A - iu}\right]^{i} I_{\frac{r_B}{C}}(\ell, i+1)$$

for $n \leq \ell$ and $m \leq k$

$$P(A)\,\psi_A(u) = \sum_{i=1}^{k} \sum_{j=0}^{\ell-1} p_{Ai} \prod_{\nu=0}^{i-1} q_{A\nu} \prod_{\xi=0}^{j} q_{B\xi} \left(\frac{r_A}{C}\right)^{i} \left(\frac{r_B}{C}\right)^{j} \binom{i+j-1}{i-1}$$

$$+ \sum_{i=1}^{\ell} p_{Ai} \prod_{\nu=0}^{i-1} q_{A\nu} \prod_{j=0}^{\ell} q_{Bj} \left(\frac{r_A}{r_A - iu}\right)^{i} I_{\frac{r_B}{C}}(\ell, i)$$

$P(AB) \; \psi_{AB}(u)$

$$= \prod_{i=0}^{k} q_{Ai} \prod_{j=0}^{\ell} q_{Bj} \left[\left(\frac{r_A}{r_A - iu} \right)^k I_{\frac{r_B}{C}}(\ell,k) + \left(\frac{r_B}{r_B - iu} \right)^\ell I_{\frac{r_A}{C}}(k,\ell) \right]$$

$$P(A) = \sum_{i=1}^{k} \sum_{j=0}^{\ell-1} p_{Ai} \prod_{\nu=0}^{i-1} q_{A\nu} \prod_{\xi=0}^{j} q_{B\xi} \left(\frac{r_A}{r_A + r_B} \right)^i \left(\frac{r_B}{r_A + r_B} \right)^j \binom{i + j - 1}{i - 1}$$

$$+ \sum_{i=1}^{k} p_{Ai} \prod_{\nu=0}^{i-1} q_{A\nu} \prod_{j=0}^{\ell} q_{Bj} \; I_{\frac{r_B}{r_A + r_B}}(\ell,i)$$

$$P(AB) = \prod_{i=0}^{k} q_{Ai} \prod_{j=0}^{\ell} q_{Bj} \; .$$

b. <u>Only A Has Limited Ammunition; A Has k Rounds, B Has</u>
<u>Unlimited Ammunition</u>

Let $\quad X_A \sim \text{ned}(r_A) \quad$ and $\quad X_B \sim \text{ned}(r_B)$.

$$P(A) = \sum_{i=1}^{k} \sum_{j=0}^{\infty} p_{Ai} \prod_{\nu=0}^{i-1} q_{A\nu} \prod_{\xi=0}^{j} q_{B\xi} \left[\frac{r_A}{r_A + r_B} \right]^i \left(\frac{r_B}{r_A + r_B} \right)^j \binom{i + j - 1}{j}$$

$$P(B) = \sum_{i=0}^{k} \sum_{j=1}^{\infty} p_{Bj} \prod_{\nu=0}^{i} q_{A\nu} \prod_{\xi=0}^{j-1} q_{B\xi} \left(\frac{r_A}{r_A + r_B} \right)^i \left(\frac{r_B}{r_A + r_B} \right)^j \binom{i + j - 1}{j - 1} \; .$$

c. Both Have Unlimited Ammunition

Let $\quad X_A \sim \text{ned}(r_A) \quad$ and $\quad X_B \sim \text{ned}(r_B)$.

$$g_0(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \prod_{\nu=0}^{i} q_{A\nu} \prod_{\xi=0}^{j} q_{B\xi} \frac{r_A^i r_B^j}{i! \, j!} t^{i+j} e^{-(r_A+r_B)t}$$

$$P(A) \, g_A(t) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \prod_{\nu=0}^{i-1} q_{A\nu} \prod_{\xi=0}^{j} q_{B\xi} \frac{r_A^i r_B^j}{(i-1)! \, j!} t^{i+j-1} e^{-(r_A+r_B)t}$$

and similarly for $P(B) \, g_B(t)$.

$$P(A) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} p_{Ai} \prod_{\nu=0}^{i-1} q_{Ai} \prod_{\xi=0}^{j} q_{B\xi} \left(\frac{r_A}{r_A+r_B}\right)^i \left(\frac{r_B}{r_A+r_B}\right)^j \frac{(i+j-1)!}{(i-1)! \, (j-1)!}$$

and similarly for $P(B)$.

Example 1: Let $p_{Ai} = \dfrac{1}{i+1}$, $p_{Bj} = \dfrac{1}{j+1}$; $p_{AO}, p_{BO} = 0$.

$$P(A) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \frac{r_A^i r_B^j}{(r_A+r_B)^{i+j}} \frac{\Gamma(i+j)}{(i+1)! \, (j+1)!}$$

$$= \frac{1}{2} \left\{ 1 + \frac{r_B}{r_A} \ln \frac{r_B}{r_A+r_B} - \frac{r_A}{r_B} \ln \frac{r_A}{r_A+r_B} \right\} .$$

Example 2: Let $p_{Ai} = (1 - \alpha^i)$ and $p_{Bj} = (1 - \beta^j)$.

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$$P(A) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} (1-\alpha^i) \, \alpha^{\frac{i(i-1)}{2}} \beta^{\frac{j(j+1)}{2}} \left[\frac{r_A^i r_B^j}{(r_A+r_B)^{i+j}} \right] \binom{i+j-1}{j} .$$

V. TIME-DEPENDENT HIT PROBABILITY

A. GENERAL IFT'S

Let $i = A, B$

$$p_i(t) = P[H \text{ by } i \mid \text{a firing at time } t],$$

$$q_i(t) = 1 - p_i(t)$$

cf of $q_i(t) = \Omega_i(u)$

$$\Theta_{0i}(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Omega_i(w) \, \phi_i(u - w) dw$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \Omega_i(w) \, \phi_i(u - w) \, \Theta_{0i}(u - w) dw$$

$$\Phi_i(u) = \phi_i(u) + [\phi_i(u) - 1] \, \Theta_{0i}(u) \quad ,$$

If $\Omega_i(w)$ has one (not necessarily simple) pole at $-w_{0i}$ in the lower half of the complex plane, then

$$S_i(u, w_{0i}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Omega_i(w) \, \phi_i(u - w) dw \qquad \text{(known)}$$

and

$$\Theta_{0i}(u) = \sum_{j=0}^{\infty} \prod_{k=0}^{j} S_i(u + k w_{0i}, w_{0i}) \quad .$$

$$P(A) = \frac{1}{2\pi i} \int_L \Phi_A(-u) \, \Phi_B(u) \, \frac{du}{u} \quad .$$

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<u>Example 1</u>: Let $X_A = \text{ned}(r_A)$, $x_B = \text{ned}(r_B)$

$$q_A(t) = \eta\, e^{-\rho t}, \qquad 0 < \eta < 1, \; \rho > 0, \; t \geq 0$$

$$q_B(t) = \xi\, e^{-\zeta t}, \qquad 0 < \xi < 1, \; \zeta > 0, \; t \geq 0$$

$$\Omega_A(u) = \frac{\eta}{\rho - iu}, \qquad \Omega_B(u) = \frac{\xi}{\zeta - iu}$$

$$\Phi_A(-u) = 1 - u\, e^{\eta r_A/\rho} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\eta r_A}{\rho} \right)^k}{k!\,[u - i(r_A + k\rho)]}$$

$$\Phi_B(u) = 1 - u\, e^{\xi r_B/\rho} \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{\xi r_B}{\zeta} \right)^j}{j!\,[u + i(r_B + j\zeta)]}$$

$$P(A) = 1 - \frac{1}{\rho} \left(\frac{\rho}{\eta r_A} \right)^{\left(\frac{r_A + r_B}{\rho} \right)} e^{\left(\frac{\eta r_A}{\rho} + \frac{\xi r_B}{\zeta} \right)}$$

$$\cdot \sum_{j=0}^{\infty} \frac{(-1)^j \left[\frac{\xi r_B}{\rho} \left(\frac{\rho}{\eta r_A} \right)^{\zeta/\rho} \right]^j \left(r_B + j\zeta \right) \gamma\left(\frac{r_B + j\zeta + r_A}{\rho}, \frac{\eta r_A}{\rho} \right)}{}$$

where $\gamma(x,y)$ is the Incomplete Gamma Function.

<u>Example 2</u>: Same as Example 1, except q_B is a constant.

$$P(A) = 1 - \frac{1}{\rho}\, p_B r_B \left(\frac{\rho}{\eta r_A} \right)^{\left(\frac{r_A + p_B r_B}{\rho} \right)} e^{\left(\frac{\eta r_A}{\rho} \right)}$$

$$\cdot \gamma\left(\frac{r_A + p_B r_B}{\rho}, \frac{\eta r_A}{\rho} \right).$$

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B. <u>IFT'S ned</u>

Let $X_A \sim \text{ned}(r_A)$ and $X_B \sim \text{ned}(r_B)$.

$p_A = p_A(t)$ is continuous, integrable, $0 \le p_A \le 1$, and

$$\lim_{a \to \infty} \int_0^a p_A(x)dx \to \infty .$$

Similarly for B.

$$P(A) = \int_0^\infty r_A\, p_A(t)$$

$$\cdot\, e^{[-r_A \int_0^t p_A(x)dx - r_B \int_0^t p_B(x)dx]}\, dt .$$

<u>Example (A Closing Engagement)</u>: Let

$$p_A(t) = \frac{a}{(r_s - vt)^2} , \quad 0 \le t \le t_0 , \quad \Big\} \quad a,b,r_s,t_0,v \text{ positive constants}$$

$$= \frac{a}{(r_s - vt_0)^2} , \quad t \ge t_0 , \quad \Big\} \quad a,b \le r_s^2 , -vt_0 < r_s .$$

For $p_B(t)$, replace a by b.

$$P(A) = \frac{ar_A}{ar_B + br_B} .$$

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C. <u>IFT's NON-STATIONARY POISSON</u>

Let $p_A(t)$ and $p_B(t)$ be as in Section B; and let $r_A(t)$ and $r_B(t)$

be such that $r_i(t)\Delta t + O(\Delta t) = P[\text{exactly 1 round fired in } (t, t + \Delta t)]$, $i = A, B$. Means both firing processes are non-stationary Poisson.

$$P(A) = 1 - \int_0^\infty p_B(\xi)\, r_B(\xi) e^{-\int_0^\xi [p_A(\eta)r_A(\eta) + p_B(\eta)r_B(\eta)]d\eta}\, d\xi$$

$$P(A)\, g_A(t) = p_A(t)\, r_A(t) e^{-\int_0^t [p_A(\xi)r_A(\xi) + p_B(\xi)r_B(\xi)]d\xi}$$

$P[A \text{ is alive at time } t]$

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$$= 1 - \int_0^t p_B(\xi)\, r_B(\xi) e^{-\int_0^\xi [p_A(\eta)r_A(\eta) + p_B(\eta)r_B(\eta)]d\eta}\, d\xi \quad .$$

D. **Hit-Probability a Function of IFT**

Let $\quad p_A(x_A) = P[H \mid \text{firing at IFT } x_A]$, $\qquad q_A(x_A) = 1 - p_A(x_A)$

$\qquad p_B(x_B) = P[H \mid \text{firing at IFT } x_B]$, $\qquad q_B(x_B) = 1 - p_B(x_B)$

cf $\quad q_A(x_A) = \Omega_A(u)$ $\qquad\qquad\qquad , \qquad q_B(x_B) = \Omega_B(u)$

with a fixed ammunition limitation of k for A and ℓ for B.

$$\Phi_{OA}(u) = \left[\frac{1}{2\pi} \int_{-\infty}^\infty \phi_A(u - w)\, \Omega_A(w)dw \right]^k \overset{\Delta}{=} \left[S_A(u) \right]^k ,$$

$$\Phi_{A1}(u) = \left[\phi_A(u) - S_A(u) \right] \left[\frac{1 - \Phi_{OA}(u)}{1 - S_A(u)} \right] ,$$

and similarly for B.

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$$P(A) = \frac{1}{2\pi i} \int_L \Phi_{A1}(-u)\, \Phi_{B1}(u)\, \frac{du}{u}$$

$$+ \left(\int_0^\infty q_B(x)\, f_B(x)dx \right)^l \left\{ 1 - \left[\int_0^\infty q_A(x)\, f_A(x)dx \right]^k \right\}$$

$$P(AB) = \left(\int_0^\infty q_A(x)\, f_A(x)dx \right)^k \left(\int_0^\infty q_B(x)\, f_B(x)dx \right)^l .$$

Example (Both Sides Unlimited Ammunition):

Let $\quad X_A \sim \text{ned}(r_A) \quad$ and $\quad X_B \sim \text{ned}(r_B)$,

$$q_A(x) = e^{-\rho_A x} \quad\text{and}\quad q_B(x) = e^{-\rho_B x} .$$

$$P(A) = \frac{r_A \rho_A [(r_B + \rho_B)(r_B + \rho_B + r_A + \rho_A) + r_A \rho_A - r_B \rho_B]}{(r_A + r_B)(\rho_A + \rho_B)(r_A + \rho_B)(r_B + \rho_A)}$$

$$P(AB) = 0$$

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VI. LIMITED AMMUNITION

A. AMMUNITION SUPPLY A RV

A draw occurs if both run out of ammunition. Let

$$P(I = i) = \alpha_i, \quad P(I = \infty) = \alpha_\infty \quad\text{and}\quad \alpha_\infty + \sum_{i=0}^\infty \alpha_i = 1, \quad i = 1,2,\ldots$$

$$P(J = j) = \beta_j, \quad P(J = \infty) = \beta_\infty \quad\text{and}\quad \beta_\infty + \sum_{j=0}^\infty \beta_j = 1, \quad j = 1,2,\ldots$$

$$\Phi_{A1}(u) = \frac{P_A\, \phi_A(u)}{1 - q_A \phi_A(u)} \left\{ 1 - \sum_{i=0}^\infty \alpha_i [q_A \phi_A(u)]^i \right\}$$

$$\Phi_{B1}(u) = \frac{P_B\, \phi_B(u)}{1 - q_B \phi_B(u)} \left\{ 1 - \sum_{j=0}^\infty \beta_j [q_B \phi_B(u)]^j \right\}$$

$$P(A) = \frac{1}{2} \left[1 - \sum_{i=0}^{\infty} \alpha_i \, q_A^i \right] \left[1 + \sum_{j=0}^{\infty} \beta_j \, q_B^j \right] + \frac{1}{2\pi i} (P) \int_{-\infty}^{\infty} \Phi_{A1}(-u) \, \Phi_{B1}(u) \, \frac{du}{u}$$

$$= \left[1 - \sum_{i=0}^{\infty} \alpha_i \, q_A^i \right] \sum_{j=0}^{\infty} \beta_j \, q_B^j + \frac{1}{2\pi i} \int_L \Phi_{A1}(-u) \, \Phi_{B1}(u) \, \frac{du}{u}$$

$$= 1 - \sum_{i=0}^{\infty} \alpha_i \, q_A^i + \frac{1}{2\pi i} \int_U \Phi_{A1}(-u) \, \Phi_{B1}(u) \, \frac{du}{u}$$

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$$P(AB) = \sum_{i=0}^{\infty} \alpha_i \, q_A^i \sum_{j=0}^{\infty} \beta_j \, q_B^j$$

$$P(N_A = n \,|\, A) = \frac{p_A \, q_A^{n-1}}{P(A)} \left(\alpha_{\infty} + \sum_{i=n}^{\infty} \alpha_i \right)$$

$$\cdot \left[\frac{1}{2} \left(1 + \sum_{j=0}^{\infty} \beta_j \, q_B^j \right) - \frac{1}{2\pi i} (P) \int_{-\infty}^{\infty} \frac{\phi_A^n(u) \, \Phi_{B1}(-u) du}{u} \right]$$

$$= \frac{p_A \, q_A^{n-1}}{P(A)} \left(\alpha_{\infty} + \sum_{i=n}^{\infty} \alpha_i \right) \left[1 - \frac{1}{2\pi i} \int_L \frac{\phi_A^n(u) \, \Phi_{B1}(-u) du}{u} \right] =$$

$$= \frac{P_A \, q_A^{n-1}}{P(A)} \left(\alpha_\infty + \sum_{i=n}^{\infty} \alpha_i \right)$$

$$\cdot \left[\sum_{j=0}^{\infty} \beta_j \, q_B^j - \frac{1}{2\pi i} \int_U \frac{\phi_A^n(u) \, \Phi_{B1}(-u) du}{u} \right] , \qquad n \geq 1$$

$$E(N_A \mid A) = \frac{1}{P(A)} \left[\frac{1}{2} \left(1 + \sum_{j=0}^{\infty} \beta_j \, q_B^j \right) S_1 - \frac{P_A}{2\pi i} (P) \int_{-\infty}^{\infty} I_1(u) du \right]$$

$$= \frac{1}{P(A)} \left[S_1 - \frac{P_A}{2\pi i} \int_L I_1(u) du \right]$$

$$= \frac{1}{P(A)} \left[\sum_{j=0}^{\infty} \beta_j \, q_B^j \, S_1 - \frac{P_A}{2\pi i} \int_U I_1(u) du \right]$$

where

$$S_1 = \left(\frac{1 - \sum\limits_{i=0}^{\infty} \alpha_i \, q_A^i}{P_A} - \sum_{i=1}^{\infty} i \, \alpha_i \, q_A^i \right) ,$$

$$I_1(u) = \frac{\phi_A(u) \, \Phi_{B1}(-u)}{[1 - q_A \phi_A(u)] u} \left[\frac{1 - \sum\limits_{i=0}^{\infty} \alpha_i (q_A \phi_A(u))^i}{1 - q_A \phi_A(u)} - \sum_{i=1}^{\infty} i \, \alpha_i (q_A \phi_A(u))^i \right] \cdot \Bigg\}$$

$$E(N_A^2 \mid A) = \frac{1}{P(A)} \left\{ \frac{1}{2} \left(1 + \sum_{j=0}^{\infty} \beta_j \, q_B^j \right) S_2 - \frac{P_A}{2\pi i} (P) \int_{-\infty}^{\infty} I_2(u) du \right\} =$$

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$$= \frac{1}{P(A)} \left\{ S_2 - \frac{P_A}{2\pi i} \int_L I_2(u)\,du \right\}$$

$$= \frac{1}{P(A)} \left\{ \sum_{j=0}^{\infty} \beta_j q_B^j S_2 - \frac{P_A}{2\pi i} \int_L I_2(u)\,du \right\}$$

where

$$S_2 = \left[\frac{(1+q_A)}{P_A^2} \left(1 - \sum_{i=0}^{\infty} \alpha_i q_A^i \right) - \frac{2}{P_A} \sum_{i=1}^{\infty} i\,\alpha_i q_A^i - \sum_{i=1}^{\infty} i^2 \alpha_i q_A^i \right] \Bigg\}$$

$$I_2(u) = \frac{\phi_A(u)\,\phi_{B1}(-u)}{[1 - q_A\phi_A(u)]u} \left\{ \frac{[1 + q_A\phi_A(u)]}{[1 - q_A\phi_A(u)]^2} \left[1 - \sum_{i=0}^{\infty} \alpha_i(q_A\phi_A(u))^i \right] \right.$$

$$\left. - \frac{2}{1 - q_A\phi_A(u)} \sum_{i=1}^{\infty} i\,\alpha_i(q_A\phi_A(u))^i - \sum_{i=1}^{\infty} i^2 \alpha_i(q_A\phi_A(u))^i \right\}$$

$$P(N_A \geq n_0 | A) = \frac{1}{P(A)} \left\{ \frac{1}{2} \left(1 + \sum_{j=0}^{\infty} \beta_j q_B^j \right) S_3 - \frac{P_A}{2\pi i} (P) \int_{-\infty}^{\infty} I_3(u)\,du \right\}$$

$$= \frac{1}{P(A)} \left\{ S_3 - \frac{P_A}{2\pi i} \int_L I_3(u)\,du \right\}$$

$$= \frac{1}{P(A)} \left\{ \sum_{j=0}^{\infty} \beta_j q_B^j S_3 - \frac{P_A}{2\pi i} \int_U I_3(u)\,du \right\}$$

where

$$S_3 = \left[\alpha_\infty + \sum_{i=0}^{\infty} \alpha_{n_0+i} \left(1 - q_A^{i+1} \right) \right] q_A^{n_0-1}$$

$$I_3(u) = \frac{\phi_A(u)\phi_{B1}(-u)}{[1 - q_A\phi_A(u)]u} \left\{ \alpha_\infty + \sum_{i=0}^{\infty} \alpha_{n_0+i} \left[1 - (q_A\phi_A(u))^{i+1} \right] \right\}$$

$$\cdot \; (q_A\phi_A(u))^{n_0-1}$$

$$P(N_A = n|AB) = \frac{\alpha_n q_A^n \sum_{j=0}^{\infty} \beta_j q_B^j}{P(AB)} \; , \qquad n \geq 0$$

$$P(N_A = n|B) = \frac{1}{P(B)} \left\{ 1 - P(N_A = 0|AB) \, P(AB) - \frac{P(N_A = 1|A) \, P(A)}{P_A} \right\} , \qquad n = 0$$

$$= \frac{1}{P_A \, P(B)} \left\{ q_A \, P(N_A = n|A) \, P(A) - P(N_A = n+1|A) \, P(A) \right.$$

$$\left. - P_A \, P(N_A = n|AB) \, P(AB) \right\} , \qquad n \geq 1$$

$$P(N_A = n) = 1 - \frac{P(A)}{P_A} \, P(N_A = 1|A) \qquad\qquad , \qquad\qquad n = 0$$

$$= \frac{P(A)}{P_A} \left[P(N_A = n|A) - P(N_A = n+1|A) \right] , \qquad\qquad n \geq 1 \qquad \text{A \& G1}$$

<u>Example 1</u>: Let $\quad X_A \sim \text{ned}(r_A) \quad$ and $\quad X_B \sim \text{ned}(r_B) \quad$,

$$\alpha_i = (1-\alpha)\alpha^i, \; \alpha_\infty = 0 \, ; \; \beta_j = (1-\beta)\beta^j, \; \beta_\infty = 0 \; .$$

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$$P(A) = \frac{\alpha p_A}{1 - \alpha q_A} \left[\frac{(1 - \alpha q_A)r_A + (1 - \beta)r_B}{(1 - \alpha q_A)r_A + (1 - \beta q_B)r_B} \right]$$

$$P(AB) = \frac{(1 - \alpha)(1 - \beta)}{(1 - \alpha q_A)(1 - \beta q_B)} \quad .$$

Example 2: Let $X_A \sim \text{ned}(r_A)$ and $X_B \sim \text{ned}(r_B)$,

$$\alpha_i = \frac{e^{-\alpha} \alpha^i}{i!}, \ \alpha_\infty = 0 ; \ \beta_j(1-\beta)\beta^j , \ \beta_\infty = 0 \quad .$$

$$P(A) = \frac{\beta p_A p_B r_A \left\{ 1 - \exp-\left(\alpha \left[\frac{p_A r_A + (1 - \beta q_B)r_B}{r_A + (1 - \beta q_B)r_B} \right] \right) \right\}}{(1 - \beta q_B)[p_A r_A + (1 - \beta q_B)r_B]} + \left(\frac{1 - \beta}{1 - \beta q_B} \right)\left(1 - e^{-\alpha p_A} \right)$$

$$P(AB) = \frac{(1 - \beta)e^{-\alpha p_A}}{1 - \beta q_B} \quad .$$

Example 3: Let $X_A \sim \text{ned}(r_A)$ and $X_B \sim \text{ned}(r_B)$,

$$\alpha_i = \left(\frac{1}{1 + \alpha} \right)^k \left(\begin{array}{c} k \\ i \end{array} \right)\alpha^i, \ \alpha_\infty = 0 ; \ i = 0,1,\ldots,k \ , \ \cdot$$

$$\beta_j = (1 - \beta)\beta^j , \quad \beta_\infty = 0 \qquad .$$

$$P(A) = \frac{\beta p_A p_B r_A}{[p_A r_A + (1 - \beta q_B)r_B](1 - \beta q_B)}$$

$$\cdot \left\{ 1 - \frac{1}{(1+\alpha)^k}\left[1 + \frac{\alpha q_A r_A}{r_A + (1 - \beta q_B)r_B} \right]^k \right\} + \left(\frac{1 - \beta}{1 - \beta q_B} \right)\left[1 - \left(\frac{1 + \alpha q_A}{1 + \alpha} \right)^k \right]$$

$$F(AB) = \left(\frac{1 + \alpha q_A}{1 + \alpha} \right)^k \frac{(1 - \beta)}{(1 - \beta q_B)} \ .$$

Example 4: Let $X_A \sim \text{Erlang}(2, r_A)$ and $X_B \sim \text{Erlang}(2, r_B)$,

$$\alpha_i = (1 - \alpha)\alpha^i, \ \alpha_\infty = 0 ; \ \beta_j = (1 - \beta)\beta^j, \ \beta_\infty = 0 \ .$$

$$P(A) = \frac{\alpha\beta \ p_A p_B \ r_A^2}{1 - \beta q_B}$$

$$\cdot \left\{ \frac{(1 - \alpha q_A)r_A^2 - (1 - \beta q_B)r_B^2 + 4r_B(r_A + r_B)}{[(1 - \alpha q_A)r_A^2 - (1 - \beta q_B)r_B^2]^2 + 4r_A r_B(r_A + r_B)[(1 - \alpha q_A)r_A + (1 - \beta q_B)r_B} \right.$$

$$+ \frac{\alpha(1 - \beta)p_A}{(1 - \alpha q_A)(1 - \beta q_B)}$$

$$P(AB) = \frac{(1 - \alpha)(1 - \beta)}{(1 - \alpha q_A)(1 - \beta q_B)} \ . \hspace{4cm} A1$$

Example 5: Let $X_A \sim \text{ned}(r_A)$ and $X_B \sim \text{ned}(r_B)$,

$$\alpha_i = (1 - \alpha_\infty)(1 - \alpha)\alpha^i$$

$$0 < \alpha, \beta < 1 \ .$$

$$\beta_j = (1 - \beta_\infty)(1 - \beta)\beta^j$$

$$P(A) = \frac{(1 - \beta_\infty)(1 - \beta)}{1 - \beta q_B} \left[1 - \frac{(1 - \alpha_\infty)(1 - \alpha)}{1 - \alpha q_A} \right] + \frac{\beta_\infty \ p_A r_A}{p_A r_A + p_B r_B} \ .$$

$$\cdot \left[1 - \frac{(1-\alpha)(1-\alpha_\infty)(r_A + p_B r_B)}{r_A(1 - \alpha q_A) + p_B r_B} \right] + \frac{p_A p_B \, r_A \, \beta(1 - \beta_\infty)}{(1 - \beta q_B)[p_A r_A + r_B(1 - \beta q_B)]}$$

$$\cdot \left\{ 1 - \frac{(1 - \alpha)(1 - \alpha_\infty)[r_A + r_B(1 - \beta q_B)]}{r_A(1 - \alpha q_A) + r_B(1 - \beta q_B)} \right\}$$

$$P(N_A = n \mid A) = \frac{p_A \, q_A^{\,n-1}}{P(A)} \left[\alpha_\infty + (1 - \alpha_\infty)\alpha^n \right]$$

$$\cdot \left[\frac{(1 - \beta_\infty)(1 - \beta)}{1 - \beta q_B} + \beta_\infty \left(\frac{r_A}{r_A + p_B r_B} \right)^n + \frac{\beta(1 - \beta_\infty)p_B}{1 - \beta q_B} \right.$$

$$\left. \cdot \left(\frac{r_A}{r_A + r_B(1 - \beta q_B)} \right)^n \right] \; .$$

<u>Example 6</u>: Let $X_A \sim \text{ned}(r_A)$ and $X_B \sim \text{ned}(r_B)$

$$c_\infty = 0, \quad \alpha_i = \left(\frac{1}{1 + \alpha} \right)^k \binom{k}{i} \alpha^i, \quad i = 0, 1, 2, \ldots, k, \quad \alpha > 0$$

$$= 0 \qquad , \quad i = k + 1, k + 2, \ldots$$

$$\beta_\infty = 1, \quad \beta_j = 0 \qquad , \quad j = 0, 1, 2, \ldots$$

$$P(A) = \frac{p_A \, r_A}{p_A r_A + p_B r_B} \left\{ 1 - \left[\left(\frac{1}{1 + \alpha} \right) \left(1 + \frac{\alpha \, q_A \, r_A}{r_A + p_B r_B} \right) \right]^k \right\}$$

$$P(N_A = n \mid A) = \frac{p_A}{q_A\, P(A)} \left(\frac{1}{1+\alpha} \right)^k \left(\frac{r_A q_A}{r_A + p_B r_B} \right)^n I_0(n, k-n+1) \ . \qquad A\,\&\,G1$$

B. FIXED AMMUNITION SUPPLY

Let $\quad \alpha_k = 1, \ \alpha_\infty = 0 , \quad$ and $\quad \alpha_i = 0 \quad$ for $i \neq k$,

$\quad \beta_\ell = 1, \ \beta_\infty = 0 , \quad$ and $\quad \beta_j = 0 \quad$ for $j \neq \ell$.

$$\Phi_{A1}(u) = \frac{p_A \phi_A(u)}{1 - q_A \phi_A(u)} \left[1 - \left(q_A \phi_A(u) \right)^k \right]$$

$$\Phi_{B1}(u) = \frac{p_B \phi_B(u)}{1 - q_B \phi_B(u)} \left[1 - \left(q_B \phi_B(u) \right)^\ell \right]$$

$$P(A) = \frac{1}{2} (1 - q_A^k)(1 + q_B^\ell) + \frac{1}{2\pi i} (P) \int_{-\infty}^{\infty} \Phi_{A1}(-u)\, \Phi_{B1}(u)\, \frac{du}{u}$$

$$= q_B^\ell (1 - q_A^k) + \frac{1}{2\pi i} \int_L \Phi_{A1}(-u)\, \Phi_{B1}(u)\, \frac{du}{u}$$

$$= 1 - q_A^k + \frac{1}{2\pi i} \int_U \Phi_{A1}(-u)\, \Phi_{B1}(u)\, \frac{du}{u}$$

$$P(AB) = q_A^k\, q_B^\ell \ .$$

The marginal increase $\Delta P(A)$ in $P(A)$, by increasing A's initial fixed supply from i to j, is

FD – CRIFT

$$\Delta P(A) = \sum_{n=1}^{j} P\left[N_A = n, A\right] - \sum_{n=1}^{i} P\left[N_A = n, A\right]$$

$$= P\left[N_A \geq i+1, A\right] - P\left[N_A \geq j+1, A\right] \quad \text{for} \quad \alpha_\infty = 1 \ .$$

Example 1: Let $\quad X_A \sim \text{ned}(r_A) \quad$ and $\quad X_B \sim \text{ned}(r_B)$

$$\alpha_k = 1, \alpha_\infty = 0 \quad \text{and} \quad \alpha_i = 0 \quad \text{for} \quad i \neq k$$

$$\beta_\ell = 1, \beta_\infty = 0 \quad \text{and} \quad \beta_j = 0 \quad \text{for} \quad j \neq \ell \ .$$

$$P(A) = \frac{p_A r_A}{p_A r_A + p_B r_B} \left[1 - \left(\frac{q_A r_A}{r_A + p_B r_B} \right)^k I_x(k, \ell) \right]$$

$$+ \frac{p_B r_B}{p_A r_A + p_B r_B} \left(\frac{q_B r_B}{p_A r_A + r_B} \right)^\ell I_y(\ell, k) - q_A^k q_B^\ell I_z(\ell, k)$$

$$P(AB) = q_A^k q_B^\ell \ .$$

$$P(N_A = n | A) = \frac{p_A}{q_A P(A)} \left\{ \left(\frac{q_A r_A}{r_A + p_B r_B} \right)^n I_x(n, \ell) + q_A^n q_B^\ell I_z(\ell, k) \right\}$$

where

$$x = \frac{r_A + p_B r_B}{r_A + r_B}, \quad y = \frac{p_A r_A + r_B}{r_A + r_B}, \quad z = \frac{r_B}{r_A + r_B} \ .$$

Example 2: Let $X_A \sim \text{ned}(r_A)$ and $X_B \sim \text{ned}(r_B)$

$\alpha_k = 1$, $\alpha_\infty = 0$, $\alpha_i = 0$ for $i \neq k$

$\beta_\infty = 1$, $\beta_j = 0$ for $j = 1, 2, \ldots$

$$P(A) = \frac{p_A \, r_A}{p_A r_A + p_B r_B} \left[1 - \left(\frac{q_A \, r_A}{p_B r_B + r_A} \right)^k \right]$$

$P(AB) = 0$.

Now, if $P(A)_U$ is the outcome of FD with both X_A and X_B ned, and if $P(A)$ is from Example 2, then

$$\frac{P(A)}{P(A)_U} = 1 - \left(\frac{q_A}{(p_B r_B / r_A) + 1} \right)^k = 1 - x^k \ .$$
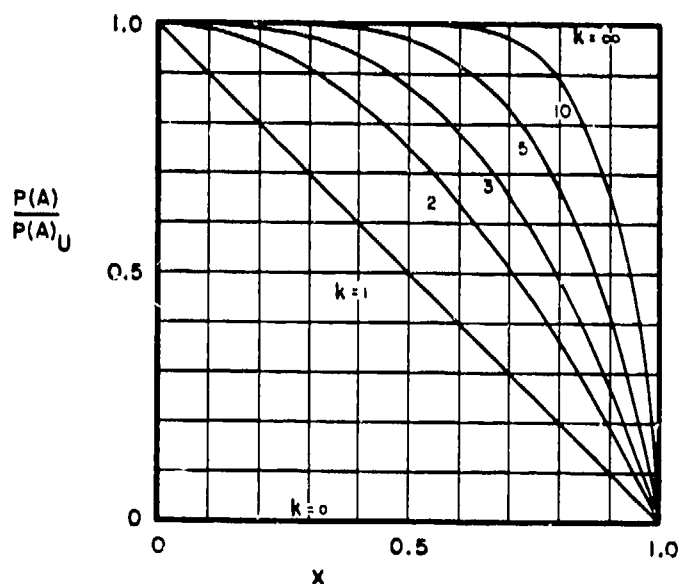
FD - CRIFT

Example 3: Let $\alpha_\infty = 1$, $\alpha_i = 0$, $i = 1,2,\ldots$
$\beta_\infty = 0$, $\beta_j = 0$, $j \neq \ell$, and $\beta_\ell = 1$

$$P(A) = \frac{p_A\,r_A}{p_A r_A + p_B r_B} + \left(\frac{p_B\,r_B}{p_A r_A + p_B r_B}\right)\left(\frac{q_B\,r_B}{p_A r_A + r_B}\right)^\ell$$

$$P(AB) = 0 .$$

. Now, if $P(A)_U$ is the outcome of FD with both X_A AND X_B ned, and if $P(A)$ is from Example 3, then

A1
$$\frac{P(A)}{P(A)_U} = 1 + \frac{p_B r_B}{p_A r_A}\left[\frac{1 - p_B}{1 + p_B\left(\frac{p_A r_A}{p_B r_B}\right)}\right]^\ell .$$

See the following page for plots of $P(A)/P(A)_U$.

C. WITHDRAWAL

A draw occurs when either contestant runs out of ammunition (the unsupplied contestant withdraws). Ammunition a rv.

$$P(A) = \frac{1}{2}\left(1 - \sum_{i=0}^{\infty}\alpha_i\,q_A^i\right)\left(1 - \sum_{j=0}^{\infty}\beta_j\,q_B^j\right)$$

$$+ \frac{1}{2\pi j}\,(P)\int_{-\infty}^{\infty}\phi_{A1}(-u)\,\phi_{B1}(u)\,\frac{du}{u} =$$

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(a)

(b)

(c)

A1
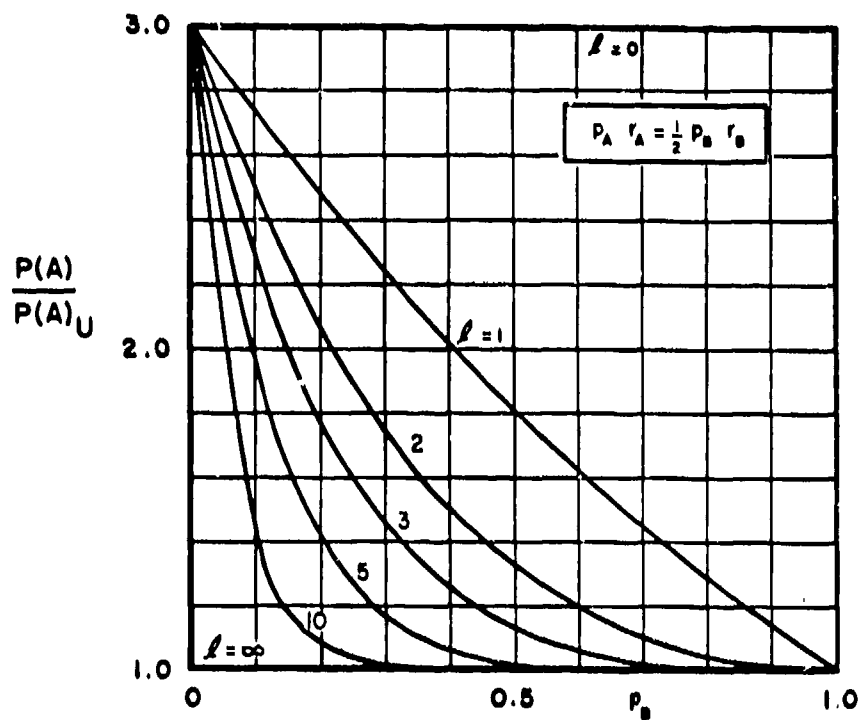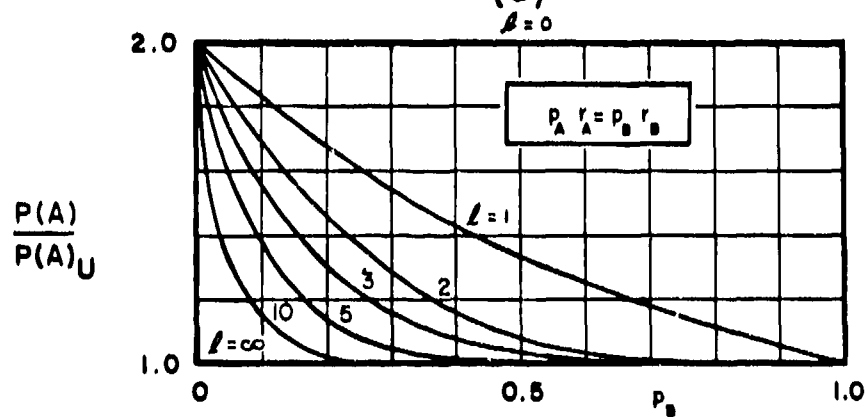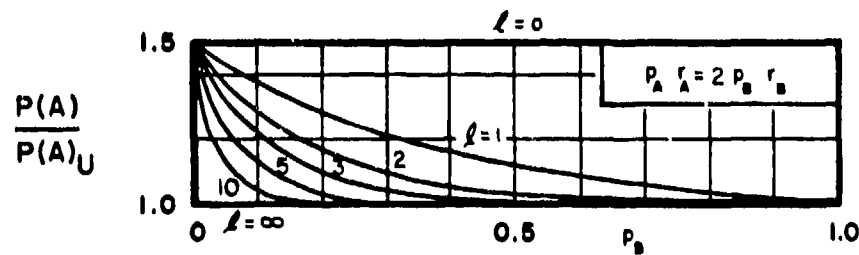
FD - CRIFT

$$= \frac{1}{2\pi i} \int_L \Phi_{A1}(-u) \, \Phi_{B1}(u) \, \frac{du}{u}$$

$$= \left(1 - \sum_{i=1}^{\infty} \alpha_i \, q_A^i \right)\left(1 - \sum_{j=0}^{\infty} \beta_j \, q_B^j \right) + \frac{1}{2\pi i} \int_U \Phi_{A1}(-u) \, \Phi_{B1}(u) \, \frac{du}{u}$$

A1 $$P(AB) = \sum_{i=0}^{\infty} \alpha_i \, q_A^i + \sum_{j=0}^{\infty} \beta_j \, q_B^j - \sum_{i=0}^{\infty} \alpha_i \, q_A^i \sum_{j=0}^{\infty} \beta_j \, q_B^j \ .$$

VII. WEAPON FAILURE (RELIABILITY) - DEPENDS ON NUMBER OF FIRINGS

T1

Failures occur only at firings. This may be interpreted as a duel with ammunition limitation.

Let $K = rv$ round number on which A has a failure

$L = rv$ round number on which B has a failure

A. NO WITHDRAWAL

Whenever a contestant discovers a failure, he cannot withdraw and remains a target.

* 1. Failures are Detected on Same Round on Which They Occur - I.E., Weapon Fires and Simultaneously Fails

Note: This is the same as the FD - CRIFT limited ammunition duel

A1   where $K = I$ and $L = J$.

2. Failures are Detected on Next Round After Failure Occurs

This is the same as running out of ammunition on the $k - 1st$ round

and discovering it at the k-th attempt to fire. <u>Note</u>: This is the same as the FD - CRIFT limited ammunition duel, where

$$K = I + 1 \quad \text{and} \quad L = J + 1 \quad .$$

B. WITHDRAWAL AFTER FAILURE

1. <u>Failures Are Detected on Same Round on Which They Occur - I.E., Failure and Detection are Simultaneous (Withdrawal is Immediate and Causes a Draw)</u>

<u>Note</u>: This is the same as the FD - CRIFT limited ammunition duel, with withdrawal, where

$$K = I \quad \text{and} \quad L = J \quad .$$ A1

2. <u>Failures Are Detected on Next Round Attempted After Failure Occurs - At Which Time Withdrawal Occurs (Causes a Draw)</u>

In this case Section B1 above cannot be adapted.

a. <u>Failure probability a constant on each round</u>

Let p_A, p_B = probability of a hit

q_A, q_B = probability of a miss

u_A, u_B = probability of a failure

then

$$p_i + q_i + u_i = 1, \quad i = A, B \quad .$$

FD - CRIFT

$$P(A) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\phi_A(-u)[\phi_B(u) - 1]du}{u} + \frac{q_B}{p_B \, 4\pi^2} \int_{-\infty}^{\infty} \frac{1}{u}$$

$$\cdot \left[\int_{-\infty}^{\infty} \frac{\phi_B(u + w)[\phi_B(w) - 1]}{w} \left(\phi_A(-u - w) - \phi_A(-w) \right) dw \right] du$$

$$P(A) = \frac{1}{2\pi i} \int_{L} \phi_A(-u) \, \phi_B(u) \, \frac{du}{u} + \frac{q_B}{p_B \, 4\pi^2} \int_{L} \frac{[\phi_B(w) - 1]}{w}$$

$$\cdot \left(\int_{U} \frac{\phi_B(u + w) \, \phi_A(-u - w) du}{u} \right) dw$$

$$P(AB) = \frac{u_A}{p_A \, 2\pi i} \int_{L} \phi_A(-u) \, \phi_B(u) \, \frac{du}{u} + \frac{u_B}{p_B \, 2\pi i} \int_{L} \phi_B(-u) \, \phi_A(u) \, \frac{du}{u}$$

$$+ \frac{u_A q_B}{p_A p_B \, 4\pi^2} \int_{L} \frac{[\phi_B(w) - 1]}{v} \left(\int_{U} \frac{\phi_B(u + w) \, \phi_A(-u - w) du}{u} \right) dw$$

T1,
A6

$$+ \frac{u_F q_A}{p_A p_B \, 4\pi^2} \int_{L} \frac{[\phi_A(w) - 1]}{w} \left(\int_{U} \frac{\phi_A(u + w) \, \phi_B(-u - w) du}{u} \right) dw .$$

<u>Example</u>: Let $X_A \sim \text{ned}(r_A)$ and $X_B \sim \text{ned}(r_E)$.

$$P(A) = \frac{p_A \, r_A}{(p_A + u_A)r_A + (p_B + u_B)r_B}$$

T1 $$P(B) = \frac{u_A r_A + u_E r_F}{(p_A + u_A)r_A + (p_B + u_B)r_B} .$$

b. **Failure a rv (function of round number)**

Let $P[K = k + 1] = \alpha_k = P[A$'s weapon fails to fire on round $k + 1]$

$P[L = \ell + 1] = \beta_\ell = P[B$'s weapon fails to fire on round $\ell + 1]$

where

$$\sum_{k=0}^{\infty} \alpha_k = \sum_{\ell=0}^{\infty} \beta_\ell = 1 \, .$$

Define N_A to be the rv, the round number on which A's weapon fails, then the geometric transform (z-transform) of N_A is

$$G_{N_A}(z) = \sum_{k=0}^{\infty} \alpha_k \, z^{k+1} \, ,$$

and

$$P(A) = \frac{1}{2\pi i} \int_L \phi_A(-u) \left[1 - \frac{G_{N_A}[q_A \phi_A(-u)]}{q_A \, \phi_A(-u)} \right] \phi_B(u) \, \frac{du}{u} + \frac{q_B}{4\pi^2} \, P_E$$

$$\cdot \int_L \frac{[\phi_B(w) - 1]}{w} \left\{ \int_U \phi_B(u + w) \left[1 - \frac{G_{N_B}[q_B \phi_B(u + w)]}{q_B \phi_B(u+w)} \right] \right\} \, .$$

$$\cdot \Phi_A(-u - w)\left[1 - \frac{G_{N_A}[q_A \phi_A(-u - w)]}{q_A \phi_A(-u - w)} \right] \frac{du}{u} \bigg\} \, dw$$

$$P(AB) = \frac{1}{2\pi i} \int_L G_{N_A}[q_A \phi_A(-u)] \, \phi_B(u) \frac{du}{u} + \frac{q_B}{4\pi^2 \, p_B} \int_L \frac{[\phi_B(w) - 1]}{w}$$

$$\cdot \left\{ \int_U \Phi_B(u + w)\left[1 - \frac{G_{N_B}[q_B \phi_B(u + w)]}{q_B \phi_B(u + w)} \right] G_{N_A}[q_A \phi_A(-u - w)] \frac{du}{u} \right\} \, dw$$

$$+ \frac{1}{2\pi i} \int_L G_{N_B}[q_B \phi_B(-u)] \, \phi_A(u) \frac{du}{u} + \frac{q_A}{4\pi^2 \, p_A} \int_L \frac{[\phi_A(w) - 1]}{w}$$

T2,
A6
$$\cdot \left\{ \int_U \Phi_A(u + w)\left[1 - \frac{G_{N_A}[q_A \phi_A(u + w)]}{q_A \phi_A(u + w)} \right] G_{N_B}[q_B \phi_B(-u - w)] \frac{du}{u} \right\} \, dw \quad .$$

VIII. LIMITED TIME-DURATION

A draw occurs if time runs out.

A. TIME LIMIT A RV

Let T_L = a time limit rv

f_{T_L} = pdf of T_L

$\Theta(u)$ = cf of $f_{T_L}(t)$.

$$P(A) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \frac{[\Theta(-u) - 1]}{u} \left(\int_{-\infty}^{\infty} \frac{\Phi_A(u - w)[\Phi_B(w) - 1]dw}{w} \right) du$$

$$= \frac{1}{4\pi^2} \int_U \frac{\Theta(-u)}{u} \left(\int_L \frac{\Phi_A(u - w)\,\Phi_B(w)dw}{w} \right) du$$

$$P(AB) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \frac{[\Phi_A(-u) - 1]}{u} \left(\int_{-\infty}^{\infty} \frac{\Theta(u - w)[\Phi_B(w) - 1]dw}{w} \right) du$$

$$= \frac{1}{4\pi^2} \int_U \frac{\Phi_A(-u)}{u} \left(\int_L \frac{\Theta(u - w)\,\Phi_B(w)dw}{w} \right) du \quad . \qquad A2$$

Let $g_A(t)$ = pdf of $T_D | A$ wins,

$\quad g_{AB}(t)$ = pdf of $T_D | AB$, (a draw)

$\quad \psi_A(u)$ = cf of $g_A(t)$ and $\psi_{AB}(u)$ = cf of $g_{AB}(t)$.

$$P(A)\, g_A(t) = \frac{1}{4\pi^2 i} \left\{ \int_{-\infty}^{\infty} e^{-iwt}\, \Phi_A(w)dw \right\} \left\{ \int_{-\infty}^{\infty} \frac{e^{-iut}[\Phi_B(u) - 1]du}{u} \right\} F^c_{T_L}(t)$$

$$= \frac{1}{4\pi^2 i} \left\{ \int_{-\infty}^{\infty} e^{-iwt}\, \Phi_A(w)dw \right\} \left\{ \int_L \frac{e^{-iut}\, \Phi_B(u)du}{u} \right\} F^c_{T_L}(t)$$

$$P(A)\, \psi_A(u) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \Phi_A(u - w) \left\{ \int_{-\infty}^{\infty} \frac{[\Phi_B(w - v) - 1][\Theta(v) - 1]dv}{v(v - w)} \right\} dw$$

$$P(AB)\, g_{AB}(t) = - \frac{f_{T_L}(t)}{4\pi^2} \left\{ \int_{-\infty}^{\infty} \frac{e^{-iut}[\Phi_A(u) - 1]du}{u} \right\} \left\{ \int_{-\infty}^{\infty} \frac{e^{-iwt}[\Phi_B(w) - 1]dw}{w} \right\} =$$

FD - CRIFT

$$\text{A \& G2} \qquad = - \frac{f_{T_L}(t)}{4\pi^2} \left\{ \int_L \frac{e^{-iut} \Phi_A(u)du}{u} \right\} \left\{ \int_L \frac{e^{-iwt} \Phi_B(w)dw}{w} \right\}$$

$$* \quad P(AB) \, \Psi_{AB}(u) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \Theta(u-w) \left\{ \int_{-\infty}^{\infty} \frac{[\Phi_A(w-v)-1][\Phi_B(v)-1]dv}{v(v-w)} \right\} dw$$

$$P(A) \, \mu_n(A) = \frac{1}{4\pi^2 \, i^n} \int_{-\infty}^{\infty} \Phi_A^{(n)}(-w) \left\{ \int_{-\infty}^{\infty} \frac{[\Phi_B(w-v)-1][\Theta(v)-1]dv}{v(v-w)} \right\} dw$$

$$= \frac{1}{4\pi^2 \, i^n} \int_{-\infty}^{\infty} \Phi_A^{(n)}(-w) \left\{ \int_U \frac{\dot\Phi_B(w-v)[\Theta(v)-1]dv}{v(v-w)} \right\} dw$$

$$= \frac{1}{4\pi^2 \, i^n} \int_{-\infty}^{\infty} \Phi_A^{(n)}(-w) \left\{ \int_L \frac{\Phi_B(w-v)-1]\Theta(v)dv}{v(v-w)} \right\} dw$$

where

$$\Phi^{(n)}(u) = \frac{d^n \Phi(u)}{du^n} \quad.$$

$$P(AB) \, \mu_n(AB) = \frac{1}{4\pi^2 \, i^n} \int_{-\infty}^{\infty} \Theta^{(n)}(-w) \left\{ \int_{-\infty}^{\infty} \frac{[\Phi_A(w-v)-1][\Phi_B(v)-1]dv}{v(v-w)} \right\} dw$$

$$= \frac{1}{4\pi^2 \, i^n} \int_{-\infty}^{\infty} \Theta^{(n)}(-w) \left\{ \int_L \frac{[\Phi_A(w-v)-1]\Phi_B(v)dv}{v(v-w)} \right\} dw$$

$$= \frac{1}{4\pi^2 \, i^n} \int_{-\infty}^{\infty} \Theta^{(n)}(-w) \left\{ \int_U \frac{\Phi_A(w-v)[\Phi_B(v)-1]dv}{v(v-w)} \right\} dw$$

where

$$\text{A \& G2} \qquad \Theta^{(n)}(u) = \frac{d^n \Theta(u)}{du^n} \quad.$$

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Example 1: Let $X_A \sim \text{ned}(r_A)$, $X_B \sim \text{ned}(r_B)$ and $f_{T_L}(t) = \frac{1}{\tau} e^{-t/\tau}$.

$$P(A) = \frac{p_A r_A}{p_A r_A + p_B r_B + \frac{1}{\tau}} \quad \text{and} \quad p(AB) = \frac{1/\tau}{p_A r_A + p_B r_B + \frac{1}{\tau}} \qquad A2$$

$$g_A(t) = g_{AB}(t) = \left(p_A r_A + p_B r_B + \frac{1}{\tau} \right) e^{-(p_A r_A + p_B r_B + \frac{1}{\tau})t} \qquad . \qquad A \& G2$$

Example 2: Let $X_A \sim \text{Erlang}(2, r_A)$. $X_B \sim \text{Erlang}(2, r_B)$ and $f_{T_L}(t) = \frac{1}{\tau} e^{-t/\tau}$

$$P(A) = p_A r_A^2 \left\{ \frac{\left(p_A r_A^2 - p_B r_B^2 \right) + 4 r_B \left(r_A + r_B \right) + \frac{1}{\tau}\left(r_A + 2r_B + \frac{1}{4\tau} \right)}{\left[p_A r_A^2 + p_B r_B^2 + 2 r_A r_B + \frac{1}{\tau}\left(r_A + r_B + \frac{1}{4\tau} \right) \right]^2 - 4 q_A q_B \, r_A^2 r_B^2} \right\}$$

$$P(AB) = \frac{1}{\tau} \frac{p_A r_A^2 \left(r_A + \frac{1}{4\tau} \right) - p_B r_B^2 \left(r_B + \frac{1}{4\tau} \right) + \left(r_A + r_B + \frac{1}{4\tau} \right)\left(4 r_A r_B + \frac{r_A + r_B}{\tau} + \frac{1}{4\tau^2} \right)}{\left[p_A r_A^2 + p_B r_B^2 + 2 r_A r_B + \frac{1}{\tau}\left(r_A + r_B + \frac{1}{4\tau} \right) \right]^2 - 4 q_A q_B \, r_A^2 r_B^2} \qquad A2$$

$$P(A) \, g_A(t) = \frac{2 p_A r_A}{\sqrt{q_A q_B}} e^{-[2(r_A + r_B) + (1/\tau)]t} \qquad \sinh 2 r_A \sqrt{q_A} \; t$$

$$\cdot \left(\sinh 2 r_B \sqrt{q_B} \; t + \sqrt{q_B} \cosh 2 r_B \sqrt{q_B} \; t \right)$$

$$P(A) \, \mu_1(A) = \frac{4 p_A r_A^2}{\sqrt{q_B}} \left[\frac{[2 r_A + 2 r_B(1 - \sqrt{q_B}) + \frac{1}{\tau}](1 + \sqrt{q_B})}{\{ [2 r_A + 2 r_B(1 - \sqrt{q_B}) + \frac{1}{\tau}]^2 - 4 r_A^2 q_A \}^2} \right. -$$

FD - CRIFT

A & G2
$$- \frac{[2r_A + 2r_B(1 + \sqrt{q_B}) + \frac{1}{\tau}](1 - \sqrt{q_B})}{\{[2r_A + 2r_B(1 + \sqrt{q_B}) + \frac{1}{\tau}]^2 - 4r_A^2 q_A\}^2}]$$

B. FIXED TIME LIMIT

Let $T_L = \tau$, a fixed number.

A2
$$P(A) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \frac{(e^{-i\tau u} - 1)}{u} \left(\int_{-\infty}^{\infty} \frac{\Phi_A(u - w)[\Phi_B(w) - 1]dw}{w} \right) du$$

*
$$= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \frac{(e^{-i\tau u} - 1)}{u} \left(\int_L \frac{\Phi_A(u - w)\Phi_B(w)dw}{w} \right) du$$

$$P(AB) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \frac{[\Phi_A(-u) - 1]}{u} \left(\int_{-\infty}^{\infty} \frac{e^{i\tau(u - w)}[\Phi_B(w) - 1]dw}{w} \right) du$$

A2
$$= \frac{1}{4\pi^2} \int_U \frac{e^{i\tau u}\Phi_A(-u)du}{u} \int_L \frac{e^{-i\tau w}\Phi_B(w)dw}{w}$$

$$P(A)g_A(t) = \frac{1}{4\pi^2 i} \left\{ \int_{-\infty}^{\infty} e^{-iwt}\Phi_A(w)dw \right\} \left\{ \int_{-\infty}^{\infty} \frac{e^{-iut}[\Phi_B(u) - 1]du}{u} \right\}$$

A & G2
$$= \frac{1}{4\pi^2 i} \left\{ \int_{-\infty}^{\infty} e^{-iwt}\Phi_A(w)dw \right\} \left\{ \int_L \frac{e^{-iut}\Phi_B(u)du}{u} \right\}, \quad t \leq \tau$$

*
$$P(A)\psi_A(u) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \Phi_A(u - w) \left\{ \int_{-\infty}^{\infty} \frac{\Phi_B(w - v) - 1][e^{iv\tau} - 1]dv}{v(v - w)} \right\} dw$$

$$P(AB)g_{AB}(t) = -\frac{\delta(t - \tau)}{4\pi^2} \left\{ \int_{-\infty}^{\infty} \frac{e^{-iut}[\Phi_A(u) - 1]du}{u} \right\} \left\{ \int_{-\infty}^{\infty} \frac{e^{-iwt}[\Phi_B(w) - 1]dw}{w} \right\} =$$

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$$= -\frac{\delta(t-\tau)}{4\pi^2}\left\{\int_L \frac{e^{-iut}\,\Phi_A(u)du}{u}\right\}\left\{\int_L \frac{e^{-iwt}\,\Phi_B(w)dw}{w}\right\},\quad t \leq \tau \quad \text{A \& G2}$$

$$P(AB)\,\psi_{AB}(u) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} e^{i(u-w)\tau}\left\{\int_{-\infty}^{\infty}\frac{[\Phi_A(w-v)-1][\Phi_B(v)-1]dv}{v(v-w)}\right\}dw \qquad *$$

$$P(A)\,\mu_n(A) = \frac{1}{4\pi^2\,i^n}\int_{-\infty}^{\infty}\Phi_A^{(n)}(-w)\left\{\int_{-\infty}^{\infty}\frac{[\Phi_B(w-v)-1][e^{iv\tau}-1]dv}{v(v-w)}\right\}dw$$

$$= \frac{1}{4\pi^2\,i^n}\int_{-\infty}^{\infty}\Phi_A^{(n)}(-w)\left\{\int_U\frac{\Phi_B(w-v)[e^{-iv\tau}-1]dv}{v(v-w)}\right\}dw$$

where

$$\Phi_A^{(n)}(u) = \frac{d^n\,\Phi_A(u)}{du^n}\,.$$

$$P(AB)\,\mu_n(AB) = \frac{\tau^n}{4\pi^2}\int_{-\infty}^{\infty}e^{-iw\tau}\left\{\int_{-\infty}^{\infty}\frac{[\Phi_A(w-v)-1][\Phi_B(v)-1]dv}{v(v-w)}\right\}dw$$

$$= \frac{\tau^n}{4\pi^2}\int_{-\infty}^{\infty}e^{-iw\tau}\left\{\int_U\frac{\Phi_A(w-v)[\Phi_B(v)-1]dv}{v(v-w)}\right\}dw$$

$$= \frac{\tau^n}{4\pi^2}\int_{-\infty}^{\infty}e^{-iw\tau}\left\{\int_L\frac{[\Phi_A(w-v)-1]\,\Phi_B(v)dv}{v(v-w)}\right\}dw\,.$$

Let $g(t)$ = pdf of T_D, then

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$$g(t) = g_A(t) \, P(A) + g_B(t) \, P(B) + g_{AB}(t) \, P(AB)$$

$$= h_A(t) \, H_B^c(t) \, F_{T_L}^c(t) + h_B(t) \, H_A^c(t) \, F_{T_L}^c(t) + f_{T_L}(t) \, H_A^c(t) \, H_B^c(t)$$

where

$$F_{T_L}^c(t) = 1 , \qquad t \leq \tau ,$$

$$= 0 , \qquad t > \tau .$$

Example 1: Let $X_A \sim \text{ned}(r_A)$ and $X_B \sim \text{ned}(r_B)$.

$$P(A) = \frac{p_A r_A}{p_A r_A + p_B r_B} \left[1 - e^{-(p_A r_A + p_B r_B)\tau} \right] ,$$

$$P(AB) = e^{-(p_A r_A + p_B r_B)\tau} \qquad .$$

$$g_A(t) = \frac{(p_A r_A + p_B r_B) e^{-(p_A r_A + p_B r_B)t}}{1 - e^{-(p_A r_A + p_B r_B)\tau}} , \qquad 0 \leq t \leq \tau$$

$$= 0 \qquad\qquad t > \tau$$

$$g_{AB}(t) = \delta(t - \tau) .$$

Example 2: Let $X_A \sim \text{Erlang}(2, r_A)$ and $X_B \sim \text{Erlang}(2, r_B)$.

$$P(A) = P(A)_f \left[1 - \frac{\exp[-2\tau(r_A + r_B)]}{\sqrt{q_A q_B}} \left(\sinh \alpha + \sqrt{q_A} \cosh \alpha \right) \left(\sinh \beta + \sqrt{q_B} \cosh \beta \right) \right]$$

$$- \frac{p_A r_A \, p_B r_B \, \exp[-2\tau(r_A + r_B)]}{\sqrt{q_A q_B} \, [(p_A r_A^2 - p_B r_B^2)^2 + 4 r_A r_B (r_A + r_B)(p_A r_A + p_B r_B)]} \, F(\alpha, \beta)$$

where

$$F(\alpha, \beta) = (p_A r_A^2 - p_B r_B^2 - 2 r_A^2 + 2 r_B^2) \sinh \alpha \sinh \beta$$

$$+ 2 \sqrt{q_B} r_B \, (r_A + r_B) \sinh \alpha \cosh \beta$$

$$- 2 \sqrt{q_A} r_A \, (r_A + r_B) \cosh \alpha \sinh \beta \quad,$$

for

$$\alpha = 2\tau \, r_A \sqrt{q_A} \quad \text{and} \quad \beta = 2\tau \, r_B \sqrt{q_B}$$

and

$$P(A)_f = p_A r_A^2 \left[\frac{(p_A r_A^2 - p_B r_B^2) + 4 r_B (r_A + r_B)}{(p_A r_A^2 - p_B r_B^2)^2 + 4 r_A r_B (r_A + r_B)(p_A r_A + p_B r_B)} \right]$$

$$P(AB) = \frac{\exp[-2\tau(r_A + r_B)]}{\sqrt{q_A q_B}} \left[\sinh \alpha + \sqrt{q_A} \cosh \alpha \right] \left[\sinh \beta + \sqrt{q_B} \cosh \beta \right] \quad \text{/\hspace{-0.3em}\cdot}$$

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$$P(A) \; g_A(t) \; = \; \frac{2p_A r_A}{\sqrt{q_A q_B}} \; e^{-[2(r_A + r_B)]t} \; \sinh 2r_A \sqrt{q_A}$$

$$\cdot \; t \left(\sinh 2r_B \sqrt{q_B} \, t + \sqrt{q_B} \cosh 2r_B \sqrt{q_B} \, t \right), \quad 0 \le t \le \tau$$

$$= \qquad\qquad 0 \qquad\qquad\quad , \quad \text{otherwise}$$

$$P(A) \; \mu_1(A) = \frac{p_A p_B r_A}{8 \sqrt{q_A q_B}} \left[-\frac{1 - (1 + 2\alpha_1 \tau)\exp(-2\alpha_1 \tau)}{(1 - \sqrt{q_b})\alpha_1^2} + \frac{1 - (1 + 2\alpha_2 \tau)\exp(-2\alpha_2 \tau)}{(1 + \sqrt{q_B})\alpha_2^2} \right.$$

$$\left. + \frac{1 - (1 + 2\alpha_3 \tau)\exp(-2\alpha_3 \tau)}{(1 - \sqrt{q_B})\alpha_3^2} - \frac{1 - (1 + 2\alpha_4 \tau)\exp(-2\alpha_4 \tau)}{(1 + \sqrt{q_B})\alpha_4^2} \right]$$

where

$$\alpha_1 \; = \; r_A(1 + \sqrt{q_A}) + r_B(1 - \sqrt{q_B}) \; ,$$

$$\alpha_2 \; = \; r_A(1 + \sqrt{q_A}) + r_B(1 + \sqrt{q_B}) \; ,$$

$$\alpha_3 \; = \; r_A(1 - \sqrt{q_A}) + r_B(1 - \sqrt{q_B}) \; ,$$

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$$\alpha_4 \; = \; r_A(1 - \sqrt{q_A}) + r_B(1 + \sqrt{q_B}) \; .$$

Example 3:

(a) Let $P(A)_U$ = outcome of FD with $X_A \sim \text{ned}(r_A)$, $X_B \sim \text{ned}(r_B)$

$P(A)$ = outcome of Example 1, T_L a rv, above,

then

$$\frac{P(A)}{P(A)_U} = \frac{\tau(p_A r_A + p_B r_B)}{\tau(p_A r_A + p_B r_B) + 1} \quad \text{(see dotted curve below)}$$

(b) Let $P(A)_U$ be as (a) above

$P(A)$ = outcome of Example 1 above where T_L is a constant, τ

then

$$\frac{P(A)}{P(A)_U} = 1 - e^{-(p_A r_A + p_B r_B)\tau} \quad \text{(see solid curve below)}$$
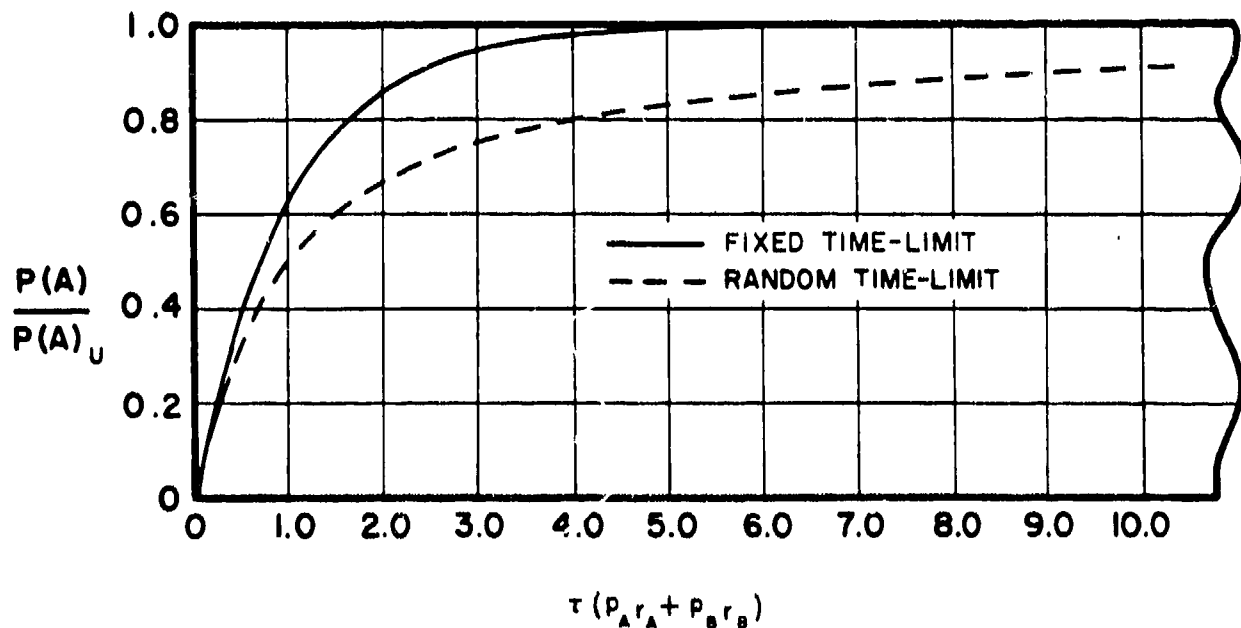
The Effect of Time-Limitation on the Outcome of a
Random Firing-Time Duel

A2

IX. TIME-RELIABILITY OF WEAPONS

This may also be interpreted as the duel with time-limitation where each side has a different limitation. X_A and X_B are rv's, and T_{LA} = rv reliability time for A, i.e., time-to-failure (time limit)

$h_{LA}(t)$ = pdf of T_{LA} Similarly for B.

$\Theta_A(u)$ = cf of $h_{LA}(t)$

A. NO WITHDRAWAL

When a contestant's weapon fails, he cannot withdraw and remains a target.

$$P(A) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Phi_A(-u)[\Theta_A(u) - 1]}{u} \left\{ 1 - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Phi_B(-w)[\Theta_B(w) - 1]dw}{w} \right.$$

$$+ \frac{1}{8\pi^3 i} \int_{-\infty}^{\infty} \frac{1}{u} \left(\int_{-\infty}^{\infty} \frac{\Phi_B(-u - v)[\Theta_B(v) - 1]dv}{v} - 1 \right)$$

$$\cdot \left(\int_{-\infty}^{\infty} \frac{\Phi_A(u - w)[\Theta_A(w) - 1]dw}{w} \right) du$$

$$P(A) = \frac{1}{2\pi i} \int_L \frac{\Phi_A(-u) \, \Theta_A(u) du}{u} \left[1 - \frac{1}{2\pi i} \int_L \frac{\Phi_B(-w) \, \Theta_B(w) dw}{w} \right]$$

$$+ \frac{1}{8\pi^3 i} \int_U \frac{1}{u} \left(\int_L \frac{\Phi_B(-u - v) \, \Theta_B(v) dv}{v} \right)$$

$$\cdot \left(\int_L \frac{\Phi_A(u - w) \, \Theta_A(w) dw}{w} \right) du$$

$$P(AB) = -\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \frac{\Theta_A(-u)[\Phi_A(u) - 1]du}{u} \int_{-\infty}^{\infty} \frac{\Theta_B(-w)[\Phi_B(w) - 1]dw}{w}$$

$$P(AB) = -\frac{1}{4\pi^2} \int_L \frac{\Theta_A(-u)\,\Phi_A(u)du}{u} \int_L \frac{\Theta_B(-w)\,\Phi_B(w)dw}{w} \quad . \qquad \text{T1,} \atop \text{A6}$$

Example: Let $\quad X_A \sim \text{ned}(r_A) \quad$ and $\quad X_B \sim \text{ned}(r_B)$

$$T_{L_A} \sim \text{ned}(\lambda_A) \quad \text{and} \quad T_{L_B} \sim \text{ned}(\lambda_B) \quad .$$

$$P(A) = \frac{p_A r_A[\lambda_B(p_A r_A + \lambda_A + p_B r_B + \lambda_B) + p_B r_B(p_A r_A + \lambda_A)]}{(p_A r_A + \lambda_A)(p_B r_B + \lambda_B)(p_A r_A + p_B r_B + \lambda_A \lambda_B)}$$

$$P(AB) = \frac{\lambda_A \lambda_B}{(p_A r_A + \lambda_A)(p_B r_B + \lambda_B)} \quad . \qquad \text{T1}$$

B. WITHDRAWAL

1. **When a Contestant's Weapon Fails He Immediately Withdraws and the Duel Ends in a Draw**

    This duel is identical with the limited-time duel, where

$$h_L(t) = h_{LA}(t)\, H_{LB}^C(t) + h_{LB}(t)\, H_{LA}^C(t) \quad . \qquad \text{T1}$$

$$P(A) = -\frac{1}{8\pi^3 i} \int_{-\infty}^{\infty} \frac{[\Theta_A(v) - 1]}{v}$$

$$\cdot \left[\int_{-\infty}^{\infty} \frac{[\Theta_B(w - v) - 1]}{w - v} \left(\int_{-\infty}^{\infty} \frac{\Phi_A(-u)[\Phi_B(u - w) - 1]du}{u - w} \right) dw \right] dv$$

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$$P(A) = -\frac{1}{8\pi^3 i} \int_L \frac{\Theta_A(v)}{v}$$

$$\cdot \left[\int_L \frac{\Theta_B(w - v)}{w - v} \left(\int_L \frac{\Phi_A(-u)\ \Phi_B(u - w)du}{u - w} \right) dw \right] dv$$

$$P(AB) = -\frac{1}{8\pi^3 i} \int_{-\infty}^{\infty} \frac{[\Phi_B(v) - 1]}{v}$$

$$\cdot \left[\int_{-\infty}^{\infty} \frac{[\Phi_A(w - v) - 1]}{w - v} \left(\int_{-\infty}^{\infty} \frac{\Theta_A(-u)[\Theta_B(u - w) - 1]du}{u - w} \right) dw \right] dv$$

$$- \frac{1}{8\pi^3 i} \int_{-\infty}^{\infty} \frac{[\Phi_B(v) - 1]}{v}$$

$$\cdot \left[\int_{-\infty}^{\infty} \frac{[\Phi_A(w - v) - 1]}{w - v} \left(\int_{-\infty}^{\infty} \frac{\Theta_B(-u)[\Theta_A(u - w) - 1]du}{u - w} \right) dw \right] dv$$

$$P(AB) = -\frac{1}{8\pi^3 i} \int_L \frac{\Phi_B(v)}{v}$$

T1,
A6
$$\cdot \left(\int_L \frac{\Phi_A(w - v)}{w - v} \left[\int_L \frac{\Theta_A(-u)\ \Theta_B(u - w) + \Theta_B(-u)\ \Theta_A(u - w)du}{u - w} \right] dw \right) dv$$

Example: Let $X_A \sim ned(r_A)$ and $X_B \sim ned(r_B)$

$$T_{LA} \sim ned(\lambda_A) \quad \text{and} \quad T_{LB} \sim ned(\lambda_B) \quad .$$

$$P(A) = \frac{p_A \, r_A}{p_A r_A + \lambda_A + p_B r_B + \lambda_B}$$

$$P(AB) = \frac{\lambda_A + \lambda_B}{p_A r_A + \lambda_A + p_B r_B + \lambda_B} \; .$$

T2

2. When a Contestant's Weapon Fails He Withdraws When He Next Tries to Fire and Discovers a Failure

$$P(A) = - \frac{q_B}{p_B \, 16\pi^4} \int_{-\infty}^{\infty} \frac{1}{v} \left\{ \int_{-\infty}^{\infty} \frac{[\phi_B(u) - 1]}{u} \left(\int_{-\infty}^{\infty} \frac{\phi_B(v + u - \rho)[\Theta_B(\rho) - 1]d\rho}{\rho} \right) \right.$$

$$\left. \cdot \left(\frac{\Theta_A(w) - 1]}{w} \left[\phi_A(-v - u - w) - \phi_A(-u - w) \right] dw \right) du \right\} dv$$

$$+ \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \frac{[\phi_B(-u) - 1]}{u} \left(\int_{-\infty}^{\infty} \frac{\phi_A(u - w)[\Theta_A(w) - 1]dw}{w} \right) du \; .$$

$$P(A) = - \frac{q_B}{p_B 16\pi^4} \int_U \frac{1}{v} \left\{ \int_U \frac{[\phi_B(u) - 1]}{u} \left(\int_L \frac{\phi_B(v + u - \rho) \, \Theta_B(\rho)d\rho}{\rho} \right) \right.$$

$$\left. \cdot \left(\int_L \frac{\phi_A(-v - u - \rho) \, \Theta_A(w)dw}{w} \right) du \right\} dv + \frac{1}{4\pi^2} \int_U \frac{\phi_B(-u)}{u}$$

$$\cdot \left(\int_L \frac{\phi_A(u - w) \, \Theta_A(w)dw}{w} \right) du$$

$$P(AB) = 1 - P(A) - P(B) \; .$$

T1,
A6

Example: Let $X_A \sim \text{ned}(r_A)$ and $X_B \sim \text{ned}(r_B)$,

$T_{LA} \sim \text{ned}(\lambda_A)$ and $T_{LB} \sim \text{ned}(\lambda_B)$.

$$P(A) = \frac{p_A r_A (p_A r_A + \lambda_A + r_B + \lambda_B)}{(p_A r_A + \lambda_A + r_B)(p_A r_A + \lambda_A + p_B r_B + \lambda_B)}$$

T1

$$P(AB) = \frac{\lambda_A(\lambda_B + r_A)(p_A r_A + \lambda_A + r_B) + \lambda_B(\lambda_A + r_B)(p_B r_B + \lambda_B + r_A)}{(p_A r_A + \lambda_A + r_B)(p_B r_B + \lambda_B + r_A)(p_A r_A + \lambda_A + p_B r_B + \lambda_B)} \quad .$$

X.   LIMITED-TIME-DURATION AND LIMITED-AMMUNITION SUPPLY

Let $X_A \sim \text{ned}(r_A)$ and $X_B \sim \text{ned}(r_B)$

$T_L \sim \text{ned}(1/\tau)$

where A has k rounds initially and B has ℓ rounds initially. Let

$$S = r_A + r_B + \frac{1}{\tau} - iu; \quad S_1 = r_B + \frac{1}{\tau} - iu; \quad \text{and} \quad S_2 = r_A + \frac{1}{\tau} - iu \quad .$$

$$P(A) = \left(\frac{p_A r_A}{p_A r_A + p_B r_B + \frac{1}{\tau}} \right)$$

$$\cdot \left[1 - \left(\frac{q_A r_A}{p_B r_B + r_A + \frac{1}{\tau}} \right)^k I_{\left(\frac{p_B r_B + r_A + (1/\tau)}{r_A + r_B + (1/\tau)} \right)}^{(k,\ell)} \right.$$

$$\left. + \left(\frac{p_B r_B}{p_A r_A + \frac{1}{\tau}} \right) \left(\frac{q_B r_B}{p_A r_A + r_B + \frac{1}{\tau}} \right)^\ell I_{\left(\frac{p_A r_A + r_B + (1/\tau)}{r_A + r_B + (1/\tau)} \right)}^{(\ell,k)} \right]$$

$$- \frac{p_A r_A}{p_A r_A + \frac{1}{\tau}} \left(\frac{q_A r_A}{r_A + \frac{1}{\tau}} \right)^k q_B^\ell I_{\left(\frac{r_B}{r_A + r_B + (1/\tau)} \right)}^{(\ell,k)}$$

$$P(AB) = \left(\frac{\frac{1}{\tau}}{p_A r_A + p_B r_B + \frac{1}{\tau}} \right) \left[1 + \left(\frac{p_A r_A}{p_B r_B + \frac{1}{\tau}} \right) \left(\frac{q_A r_A}{p_B r_B + r_A + \frac{1}{\tau}} \right)^k \quad .$$

$$\cdot \ I_{\left(\frac{p_A r_A + p_B r_B + (1/\tau)}{r_A + r_B + (1/\tau)}\right)}(k,\ell) + \left(\frac{p_B r_B}{p_A r_A + \frac{1}{\tau}}\right)\left(\frac{q_B r_B}{p_A r_A + r_B + \frac{1}{\tau}}\right)^{\ell}$$

$$\cdot \ I_{\left(\frac{p_A r_A + p_B r_B + (1/\tau)}{r_A + r_B + (1/\tau)}\right)}(\ell,k) \Bigg] + \frac{r_A r_A q_B^{\ell}}{p_A r_A + \frac{1}{\tau}}\left(\frac{q_A r_A}{r_A + \frac{1}{\tau}}\right)^{k} I_{\left(\frac{r_B}{r_A + r_B + (1/\tau)}\right)}(\ell,k)$$

$$+ \frac{p_B r_B q_A^{k}}{p_B r_B + \frac{1}{\tau}}\left(\frac{q_B r_B}{r_B + \frac{1}{\tau}}\right)^{\ell} I_{\left(\frac{r_A}{r_A + r_B + (1/\tau)}\right)}(k,\ell)$$

$$P(A)\ \psi_A(u) = \frac{p_A r_A}{p_A r_A + p_B r_B + \frac{1}{\tau} - iu}\Bigg\{\Bigg[1 - \left(\frac{q_A r_A}{r_A + p_B r_B + \frac{1}{\tau} - iu}\right)^{k} I_{\left(\frac{p_A r_A + S_1}{S}\right)}(\ell,k)\Bigg]$$

$$+ \frac{p_B r_B}{p_A r_A + \frac{1}{\tau} - iu}\left(\frac{q_B r_B}{p_A r_A + r_B + \frac{1}{\tau} - iu}\right)^{\ell} I_{\left(\frac{p_A r_A + S_1}{S}\right)}(\ell,k)\Bigg\}$$

$$- \frac{p_A r_A}{p_A r_A + \frac{1}{\tau} - iu}\left(\frac{q_A r_A}{S_2}\right)^{k} q_B^{\ell}\ I_{(r_B/S)}(\ell,k)$$

$$P(AB)\ \psi_{AB}(u) = q_B^{\ell}\left(\frac{p_A r_A - iu}{p_A r_A + \frac{1}{\tau} - iu}\right)\left(\frac{q_A r_A}{S_2}\right)^{k} I_{(r_B/S)}(\ell,k)$$

$$+ \ q_A^{k}\left(\frac{p_B r_B - iu}{p_B r_B + \frac{1}{\tau} - iu}\right)\left(\frac{q_B r_B}{S_1}\right)^{\ell} I_{(r_A/S)}(k,\ell)\ +$$

$$+ \left(\frac{1/\tau}{p_A r_A + p_B r_B + \frac{1}{\tau} - iu} \right) \left\{ 1 + \left(\frac{p_A r_A}{p_B r_B + \frac{1}{\tau} - iu} \right) \left(\frac{q_A r_A}{p_B r_B + r_A + \frac{1}{\tau} - iu} \right)^k \right.$$

$$\cdot \; I \left(\frac{p_A r_A + p_B r_B + (1/\tau) - iu}{S} \right)^{(k,\ell)} + \left(\frac{p_B r_B}{p_A r_A + \frac{1}{\tau} - iu} \right) \left(\frac{q_B r_B}{p_A r_A + r_B + \frac{1}{\tau} - iu} \right)^\ell$$

$$\cdot \; I \left(\frac{p_A r_A + p_B r_B + (1/\tau) - iu}{S} \right)^{(\ell,k)} \left. \right\} .$$

<u>Special Case</u>: B has unlimited ammunition.

$$P(A) = \frac{p_A r_A}{p_A r_A + p_B r_B + \frac{1}{\tau}} \left[1 - \left(\frac{q_A r_A}{p_B r_B + r_A + \frac{1}{\tau}} \right)^k \right]$$

$$\text{Bh3(3)} \quad P(AB) = \frac{1/\tau}{p_A r_A + p_B r_B + \frac{1}{\tau}} \left[1 + \frac{p_A r_A}{p_B r_B + \frac{1}{\tau}} \left(\frac{q_A r_A}{p_B r_B + r_A + \frac{1}{\tau}} \right)^k \right] .$$

XI. <u>INTERRUPTED FIRING</u>

A. <u>FIRING WITH WEAPONS WHICH FAIL AND CAN BE REPAIRED (REPLACED)</u>

Each contestant fires until he hits or his weapon fails. When a weapon fails it is repaired (replaced) and firing resumes. Time-to-failure is an independent rv and time-to-repair (replace) is an independent rv. When his weapon fails, the contestant is still

vulnerable. The process continues to a kill.

1. <u>Fixed Ammunition-Limitation</u>

k rounds for A and ℓ rounds for B

rv time-to-failure for $\begin{cases} A & \text{is} & T_{LA} \sim \text{ned}(r_{LA}) \\ B & \text{is} & T_{LB} \sim \text{ned}(r_{LB}) \end{cases}$

rv time-to-repair(replace) for $\begin{cases} A & \text{is} & T_{\tilde{C}A} & \text{with} & \text{cf} & \Theta_{\tilde{C}A}(u) \\ B & \text{is} & T_{\tilde{C}B} & \text{with} & \text{cf} & \Theta_{\tilde{C}B}(u) \end{cases}$.

$$\Phi_A(u) = \frac{p_A(r_{LA} - iu)\,\phi(u + ir_{LA})}{(r_{LA} - iu)[1 - q_A\phi_A(u + ir_{LA})] - r_{LA}[1 - \phi_A(u + ir_{LA})\Theta_{\tilde{C}A}(u)]}$$

$$\cdot \left\{ 1 - \left[\frac{(r_{LA} - iu)\,\phi_A(u + ir_{LA})q_A}{(r_{LA} - iu) - r_{LA}[1 - \phi_A(u + ir_{LA})]\Theta_{\tilde{C}A}(u)} \right]^k \right\} + q_A^k$$

$$= \Phi_{A1}(u) + q_A^k$$

$$P(A) = \frac{1}{2\pi i} \int_L \Phi_{A1}(-u)\,\Phi_{B1}(u)\,\frac{du}{u} + q_B^\ell[1 - q_A^k]$$

$$P(AB) = q_A^k\, q_B^\ell \ .$$

    a.   A has unlimited ammunition $(k \to \infty)$

        B has a fixed supply of ℓ rounds

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Then

$$P(A) = \frac{1}{2\pi i} \int_L \Phi_A(-u) \Phi_{B1}(u) \frac{du}{u} + q_{\bar{B}}^{\ell} \, ,$$

and

$$P(AB) = 0.$$

Example 1: Both sides have unlimited ammunition. Let

$$X_A \sim \text{ned}(r_A) \quad \text{and} \quad X_B \sim \text{ned}(r_B) \, ,$$

$$T_{\bar{C}A} \sim \text{ned}(r_{\bar{C}A}) \quad \text{and} \quad T_{\bar{C}B} \sim \text{ned}(r_{\bar{C}B}) \, .$$

$$P(A) = \frac{1}{r_{\bar{C}A}}$$

$$\cdot \frac{\left[\begin{array}{c} r_{\bar{C}A}(p_B r_B + r_{\bar{C}B} + r_{LB})(r_{\bar{C}A} + r_{LA} + p_A r_A + r_{\bar{C}B} + r_{LB} + p_B r_B) \\ + (p_A r_A r_{\bar{C}A} - p_B r_B r_{\bar{C}B})(r_{\bar{C}A} + p_B r_B) \end{array} \right]}{\left[\begin{array}{c} (p_A r_A r_{\bar{C}A} - p_B r_B r_{\bar{C}B})^2 + (p_A r_A + r_{\bar{C}A} + r_{LA})(p_B r_B + r_{\bar{C}B} + r_{LB})(p_A r_A r_{\bar{C}A} - p_B r_B r_{\bar{C}B}) \\ + p_A r_A r_{\bar{C}A}(p_B r_B + r_{\bar{C}B} + r_{LB})^2 + p_B r_B r_{\bar{C}B}(p_A r_A + r_{\bar{C}A} + r_{LA})^2 \end{array} \right]}$$

Example 2: B is failure-free (no failures). Both sides have unlimited ammunition. Let $X_A \sim \text{ned}(r_A)$; $X_B \sim \text{ned}(r_B)$; $T_{\bar{C}A} \sim \text{ned}(r_{\bar{C}A})$.

$$P(A) = \frac{p_A r_A(r_{\bar{C}A} + p_B r_B)}{(p_B r_B)^2 + p_B r_B(p_A r_A + r_{\bar{C}A} + r_{LA}) + p_A r_A r_{\bar{C}A}} \, .$$

2. Ammunition Limitation is a RV

$$A: \quad P[I = i] = \alpha_i ; \quad \sum_{i=0}^{\infty} \alpha_i = 1$$

For

$$B: \quad P[J = j] = \beta_j ; \quad \sum_{j=0}^{\infty} \beta_j = 1$$

$$\Phi_A(u) = \frac{p_A(r_{LA} - iu) \, \phi_A(u + ir_{LA})}{(r_{LA} - iu)[1 - q_A\phi_A(u + ir_{LA})] - r_{LA}[1 - \phi_A(u + ir_{LA})]\Theta_{\bar{C}A}(u)}$$

$$\cdot \left\{ 1 - \sum_{i=0}^{\infty} \alpha_i \left[\frac{(r_{LA} - iu) \, \phi(u + ir_{LA})q_A}{(r_{LA} - iu) - r_{LA}[1 - \phi_A(u + ir_{LA})]\Theta_{\bar{C}A}(u)} \right]^i \right\} + \sum_{i=0}^{\infty} \alpha_i \, q_A^i$$

$$= \Phi_{A1}(u) + \sum_{i=0}^{\infty} \alpha_i \, q_A^i$$

$$P(A) = \frac{1}{2\pi i} \int_L \Phi_{A1}(-u) \, \Phi_B(u) \, \frac{du}{u} + \sum_{j=0}^{\infty} \beta_j \, q_B^j \left(1 - \sum_{i=0}^{\infty} \alpha_i \, q_A^i \right)$$

$$P(AB) = \sum_{i=0}^{\infty} \alpha_i \, q_A^i \sum_{j=0}^{\infty} \beta_j \, q_B^j \; . \hspace{3cm} \text{Bh3(4)}$$

B. BOTH SIDES OUT OF CONTACT PERIODICALLY

1. Three-State System

Let $X_A \sim \text{ned}(r_A)$ and $X_B \sim \text{ned}(r_B)$. The system cycles through three states repeatedly, until a kill occurs. States are:

(1) Combat (Contact), C ,

(2) No Contact, $\bar{C}$,

(3) searching, s .

Time is zero at beginning of first contact. All three states are randomly time-limited with continuous rv's, as follows:

X_C - rv time length of contact $\sim \text{ned}(r_C)$

$X_{\bar{C}}$ - rv time length of no contact - $\text{ned}(r_{\bar{C}})$

X_s - rv time length of searching $\sim$ general rv with cf of pdf as $\phi_s(u)$

T_C - rv time since start, in contact, no hit

$T_{\bar{C}}$ - rv time since start, not in contact, no hit

T_s - rv time since start, searching, no hit

y_s - rv time since last search started $|$ in search state

$\Theta_C(u)$ - cf of DF of T_C

$\Theta_{\bar{C}}(u)$ - cf of DF of $T_{\bar{C}}$

$\Theta_s(u)$ - cf of DF of T_s

$\Psi_s(u,y_s)$ - cf of joint DF of T_s and pdf of y_s

and where the superscripts k, ℓ refer to k rounds fired by A and ℓ rounds fired by B, up to time t.

Let $C_1(u) = (r_C + r_A + r_B - iu)$,

$$C_2(u) = C_1(u)(r_{\bar{C}} - iu) - r_C r_{\bar{C}} \phi_s(u) .$$

$$\Theta_C = \frac{r_{\bar{C}} - iu}{(r_{\bar{C}} - iu)(C_1(u) - r_A q_A - r_B q_B) - r_C r_{\bar{C}} \phi_s(u)}$$

$$\psi_s(u,0) = \frac{r_C r_{\bar{C}}}{r_{\bar{C}} - iu} \Theta_C(u)$$

$$\Theta_{\bar{C}}(u) = \frac{r_C}{r_{\bar{C}} - iu} \Theta_C(u)$$

$$\Theta_s(u) = \frac{r_C r_{\bar{C}}}{iu - r_{\bar{C}}} \left[\frac{1 - \phi_s(u)}{iu} \right] \Theta_C(u)$$

$$\psi_s(u,y_s) = \psi_s(0,u)e^{iuy_s - \int_0^{y_s} \lambda_s(\xi)d\xi}$$

$$\psi_A(u) = \frac{p_A r_A}{- iu} \Theta_C(u)$$

$$\psi_B(u) = \frac{p_B r_B}{- iu} \Theta_C(u)$$

$$\Theta_C^{k,\ell}(u) = \binom{k+\ell}{k} \frac{(r_{\bar{C}} - iu)^{k+\ell+1} (q_A r_A)^k (q_B r_B)^\ell}{C_2(u)^{k+\ell+1}}$$

$$\Theta_{\bar{C}}^{k,\ell}(u) = \binom{k+\ell}{k} \frac{r_C(r_{\bar{C}} - iu)^{k+\ell} (q_A r_A)^k (q_B r_B)^\ell}{C_2(u)^{k+\ell+1}}$$

$$\Theta_s^{k,\ell}(u) = \binom{k+\ell}{k} \frac{r_C r_{\bar{C}}[1 - \phi_s(u)](r_{\bar{C}} - iu)^{k+\ell} (q_A r_A)^k (q_B r_B)^\ell}{-iu[C_2(u)]^{k+\ell+1}}$$

$$\psi_A^{k,\ell}(u) \;=\; \binom{k + \ell + 1}{k - 1}\; \frac{p_A r_A (r_{\bar C} - iu)^{k+\ell} (q_A r_A)^{k-1} (q_B r_B)^{\ell}}{- iu [C_2(u)]^{k+\ell}}$$

$$\psi_B^{k,\ell}(u) \;=\; \binom{k + \ell - 1}{\ell - 1}\; \frac{p_B r_B (r_{\bar C} - iu)^{k+\ell} (q_A r_A)^{k} (q_B r_B)^{\ell-1}}{- iu [C_2(u)]^{k+\ell}}$$

<u>Example</u>: Let $X_s \sim \mathrm{ned}(r_s)$

$$C_3(t) \;=\; \frac{t^{k+\ell-m} e^{\alpha_n t}}{(k + \ell - m)(m - 1)!}$$

$$C_4(u) \;=\; \left[\frac{(r_C - iu)(r_s - iu)}{(i) \displaystyle\prod_{\xi=1,\,\xi\neq n}^{3} (u - \alpha_\xi)} \right]^{k+\ell}$$

where α_n is the n-th root of $C_5(u)$, $n = 1,2,3$ and

$$C_5(u) \;=\; iu^3 - u^2 (r_A + r_B + r_C + r_{\bar C} + r_s)$$

$$\quad - iu[r_s r_{\bar C} + (r_{\bar C} + r_s)(r_A + r_B + r_s)] + r_{\bar C} r_s (r_A + r_B)$$

$$F_C^{k,\ell}(t) \;=\; \sum_{m=1}^{k+\ell+1} \sum_{n=1}^{3} \frac{t\,[C_3(t)]}{(k + \ell + 1 - m)}\; C_6(\alpha_n) \;, \quad \text{where}$$

$$C_6(\alpha_n) \;=\; \frac{d^{m-1}}{du^{m-1}} \left[\binom{k + \ell}{\ell} \frac{(r_A q_A)^k (r_B q_B)^{\ell} C_4(u)}{(r_C - iu)(r_s - iu) \displaystyle\prod_{\xi=1,\,\xi\neq n}^{3} (u - \alpha_\xi)} \right]_{u = \alpha_n}$$

$$F_C^{k,\ell}(t) = \sum_{m=1}^{k+\ell+1} \sum_{n=1}^{3} \frac{t(C_3(t))}{(k + \ell + 1 - m)} \, C_7(\alpha_n) \, , \quad \text{where}$$

$$C_7(\alpha_n) = \frac{d^{m-1}}{du^{m-1}} \left[\binom{k + \ell}{\ell} \frac{r_C(r_A q_A)^k \, (r_B q_B)^\ell \, (r_s - iu) \, C_4(u)}{\prod\limits_{\xi=1, \xi \neq n}^{3} (u - \alpha_\xi)} \right]_{u = \alpha_n}$$

$$F_s^{k,\ell}(t) = \sum_{m=1}^{k+\ell+1} \sum_{n=1}^{3} \frac{t(C_3(t))}{(k + \ell + 1 - m)} \, C_8(\alpha_n) \, , \quad \text{where}$$

$$C_8(\alpha_n) = \frac{d^{m-1}}{du^{m-1}} \left[\binom{k + \ell}{\ell} \frac{r_C r_C^*(r_A q_A)^k \, (r_B q_B)^\ell \, C_4(u)}{\prod\limits_{\xi=1, \xi \neq n}^{3} (u - \alpha_\xi)} \right]_{u = \alpha_n}$$

$$g_A^{k,\ell}(t) = \sum_{m=1}^{k+\ell} \sum_{n=1}^{3} C_3(t) \, C_9(\alpha_n) + (-1)^{k+\ell} \binom{k + \ell - 1}{k - 1}$$

$$\cdot \; p_A r_A (q_A r_A)^{k-1} \frac{(q_B r_B)^\ell \, (r_C^* r_s)^{k+\ell}}{(\alpha_1 \alpha_2 \alpha_3)^{k+\ell}} \, , \quad \text{where}$$

$$C_9(\alpha_n) = \frac{d^{m-1}}{du^{m-1}} \left[\binom{k + \ell - 1}{k - 1} p_A r_A (q_A r_A)^{k-1} \, (q_B r_B)^\ell \, C_4(u) \right]_{u = \alpha_n}$$

$$g_B^{k,\ell}(t) = \sum_{m=1}^{k+\ell} \sum_{n=1}^{3} C_3(t) \, C_{10}(\alpha_n) + (-1)^{k+\ell} \binom{k + \ell - 1}{k - 1} \; .$$

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$$\cdot \ p_B r_B (r_A q_A)^k \ \frac{(r_B q_B)^{\ell-1} (r_{\bar{C}} r_s)^{k+\ell}}{(\alpha_1 \alpha_2 \alpha_3)^{k+\ell}} \quad , \quad \text{where}$$

Ga & Sl
$$C_{10}(\alpha_n) \ = \ \frac{d^{m-1}}{du^{m-1}} \left[\binom{k+\ell-1}{k-1} p_B r_B (r_A q_A)^k \ (r_B q_B)^{\ell-1} \ C_4(u) \right]_{u=\alpha_n} \quad .$$

2. <u>Four-State System</u>

Let $X_A \sim \text{ned}(r_A)$, $X_B \sim \text{ned}(r_B)$. The system cycles through four states repeatedly until a kill occurs. The states are:

    (1) combat (contact), C

    (2) No contact, $\bar{C}$

    (3) Searching, s

    (4) Reclosing for combat, r .

Time is zero at the beginning of the first contact (combat); all four states are randomly limited with continuous rv's defined as follows:

    X_C - general rv with cf of pdf, $\phi_C(u)$

    $X_{\bar{C}}$ - general rv with cf of pdf, $\phi_{\bar{C}}(u)$

    X_s - general rv with cf of pdf, $\phi_s(u)$

    X_r - general rv with cf of pdf, $\phi_r(u)$

and where

    T_C - time-since-start to beginning of last contact (combat) period, or last unsuccessful round fired

$T_{\bar{C}}$ - time-since-start to beginning of last no-contact period

T_s - time-since-start to beginning of last search period

T_r - time-since start to beginning of last reclosing period

T_A - time-to-kill by A

T_B - time-to-kill by B

z, w - transform variables for two-variable geometric transforms (gt) .

Let $j = C, \bar{C}, s,$ or r

$\psi_j(z, w, u, y_j) = zw$ gt of $\psi_j(u, y_j)$ which is the cf of the joint DF of T_j and the pdf of y_j (time-since event j last occurred)

$\psi_j(z, w, u) = zw$ gt of $\psi_j(u)$ where

$\psi_C(u) = $ cf of $P[T_C < t,$ time-to-next event (a firing by A, a firing by B, or end of end contact) $> t]$

$\psi_{\bar{C}}(u) = $ cf of $P[T_{\bar{C}} < t,$ time-to-end of no contact period $> t]$

$\psi_s(u) = $ cf of $P[T_s < t,$ time-to-end of search period $> t]$

$\psi_r(u) = $ cf of $P[T_r < t,$ time-to-end of reclosing period $> t]$

$\psi_A(z, w, u) = zw$ gt of $\psi_A(u)$ which is the cf of

$$P[t < T_A < t + dt \mid \text{kill by } A]$$

$\psi_B(z, w, u) = zw$ gt of $\psi_B(u)$ which is the cf of

$$P[t < T_B < t + dt \mid \text{kill by } B]$$

$$\Theta_C(z,w,u) = zw \quad \text{gt} \quad \text{of} \quad \Theta_C(u) \quad \text{which is the} \quad \text{cf} \quad \text{of}$$

$$P[t < T_C < t + dt] \quad .$$

Let $C_1 = r_A + r_B - r_A q_A z - r_B q_B z$

$$C_2(u) = 1 - \phi_C(u + iC_1) \, \phi_r(u) \, \phi_{\bar{C}}(u) \, \phi_s(u)$$

$$\Theta_C(z,w,u) = \frac{1 - C_2(u)}{C_2(u)}$$

$$\psi_C(z,w,u,y_C) = [U(y_C) + \Theta_C(z,w,u)] e^{iuy_C - C_1 y_C - \int_0^{y_C} \lambda_C(\xi)d\xi}$$

$$\psi_{\bar{C}}(z,w,u,y_{\bar{C}}) = [1 + \Theta_C(z,w,u)] \, \phi_C(u + iC_1) e^{iuy_{\bar{C}} - \int_0^{y_{\bar{C}}} \lambda_{\bar{C}}(\xi)d\xi}$$

$$\psi_s(z,w,u,y_s) = [1 + \Theta_C(z,w,u)] \, \phi_C(u + iC_1) \, \phi_{\bar{C}}(u) e^{iuy_s - \int_0^{y_s} \lambda_s(\xi)d\xi}$$

$$\psi_r(z,w,u,y_r) = \frac{\Theta_C(z,w,u)}{\phi_r(u)} \, e^{iuy_r - \int_0^{y_r} \lambda_r(\xi)d\xi}$$

$$\psi_C(z,w,u) = \frac{(1 - \phi_C(u + iC_1))}{C_2(u)(C_1 - iu)}$$

$$\psi_{\bar{C}}(z,w,u) = \frac{\phi_C(u + iC_1)}{C_2(u)} \cdot \frac{(\phi_{\bar{C}}(u) - 1)}{iu}$$

$$\psi_s(z,w,u) = \frac{\phi_C(u + iC_1) \, \phi_{\bar{C}}(u)}{C_2(u)} \cdot \frac{(\phi_s(u) - 1)}{iu}$$

$$\psi_r(z,w,u) = \frac{\phi_C(u + iC_1)\,\phi_{\bar{C}}(u)\,\phi_s(u)}{C_2(u)} \cdot \frac{(\phi_r(u) - 1)}{iu}$$

$$P(A)\,\psi_A(z,w,u) = \frac{p_A r_A z[\phi_C(u + iC_1) - 1]}{C_2(u)(iu - C_1)}$$

$$P(B)\,\psi_B(z,w,u) = \frac{p_B r_B w[\phi_C(u + iC_1) - 1]}{C_2(u)(iu - C_1)}$$

$$P(A) = P(A)\,\psi_A(1,1,0) = \frac{p_A r_A}{p_A r_A + p_B r_B}$$

N.B.:

(1) To get any $\psi(u,y)$, $\psi(u)$ or $\Theta(u)$, let $z = 1$, $w = 1$.

(2) If the cf's above are differentiated k times, with respect to z and ℓ times with respect to w, and then set $z = w = 0$, one obtains $\psi^{k,\ell}(u,y)$ or $\Theta^{k,\ell}(u)$ or $\psi^{k,\ell}(u)$, which are the cf's of the probability functions defined earlier with exactly k and ℓ rounds fired.

Sr, G & Si

3. Four-State System with Limited Ammunition

Let $X_A \sim \text{ned}(r_A)$, $X_B \sim \text{ned}(r_B)$ with the system cycles the same as in Section 2 above, except limited ammunition and $X_C \sim \text{ned}(r_C)$, $X_{\bar{C}} \sim \text{ned}(r_{\bar{C}})$, with A allowed K rounds and B, L rounds. The superscripts k,ℓ denote k rounds fired by A and ℓ rounds fired by B.

FD - CRIFT

Let

$$c_1 = (r_A + r_B + r_C - iu)$$

$$c_2 = (r_B + r_C - iu)$$

$$c_3 = (r_A + r_C - iu)$$

$$c_4 = (r_C - iu)$$

$$c_5 = [c_1(r_{\bar{C}} - iu) - r_C r_{\bar{C}} \, \phi_s(u) \, \phi_r(u)]^{-1}$$

$$c_6 = [c_2(r_{\bar{C}} - iu) - r_C r_{\bar{C}} \, \phi_s(u) \, \phi_r(u)]^{-1}$$

$$c_7 = [c_3(r_{\bar{C}} - iu) - r_C r_{\bar{C}} \, \phi_s(u) \, \phi_r(u)]^{-1}$$

$$c_8 = [c_4(r_{\bar{C}} - iu) - r_C r_{\bar{C}} \, \phi_s(u) \, \phi_r(u)]^{-1}$$

$$\psi_C^{k,\ell}(u) = \binom{\ell + k}{\ell} (q_A r_A)^k (q_B r_B)^\ell \, [(r_{\bar{C}} - iu)c_5]^{k+\ell+1} \, , \quad \begin{array}{c} 1 \le k < K \\ 1 \le \ell < L \end{array}$$

$$\psi_C^{K,\ell}(u) = (r_{\bar{C}} - iu)^{K+\ell+1} (q_A r_A)^K (q_B r_B)^\ell \sum_{j=0}^{\ell} \binom{K + j - 1}{j} c_5^{K+j} \, c_6^{\ell-j+1}$$

$$\text{for} \quad 1 \le \ell \le L-1$$

$$\psi_C^{k,L}(u) = (r_{\bar{C}} - iu)^{k+L+1} (q_A r_A)^k (q_B r_B)^L \sum_{j=0}^{k} \binom{L + j - 1}{j} c_5^{L+j} \, c_7^{k-j+1}$$

$$\text{for} \quad 1 \le k \le K-1$$

$$\psi_C^{0,L}(u) = (r_{\bar{C}} - iu)^{L+1} (q_B r_B)^L \, c_7 \, c_5^L$$

$$\psi_C^{K,0}(u) = (r_{\bar{C}} - iu)^{K+1} (q_A r_A)^K \, c_6 \, c_5^K$$

$$\psi_C^{k,0}(u) = (r_{\bar{C}} - iu)^{k+1} (q_A r_A)^k c_5^{k+1} \ , \quad 0 < k < K$$

$$\psi_C^{0,\ell}(u) = (q_B r_B)^\ell \, [(r_{\bar{C}} - iu)c_5]^{\ell+1} \ , \quad 0 < \ell < L$$

$$P(AB) \, \psi_C^{K,L}(u) = (r_{\bar{C}} - iu)^{K+L} (q_A r_A)^K (q_B r_B)^L$$

$$\cdot \left\{ c_5^L c_7^K \sum_{i=0}^{K-1} \binom{L+i-1}{i} \left(\frac{c_5}{c_7}\right)^i + c_5^K c_6^L \binom{K+j-1}{j} \left(\frac{c_5}{c_6}\right)^j \right\} \qquad **$$

$$\psi_C^{0,0}(u) = (r_{\bar{C}} - iu)c_5$$

$$\psi_{\bar{C}}^{k,\ell}(u) = \frac{r_C}{r_{\bar{C}} - iu} \, \psi_C^{k,\ell}(u)$$

$$\psi_s^{k,\ell}(u,y_s) = \frac{r_C \, r_{\bar{C}}}{r_{\bar{C}} - iu} \, \psi_C^{k,\ell}(u) e^{iuy_s - \int_0^{y_s} \lambda_s(\xi)d\xi}$$

$$\psi_s^{k,\ell}(u) = \frac{r_C \, r_{\bar{C}}}{r_{\bar{C}} - iu} \left(\frac{\phi_s(u) - 1}{iu} \right) \psi_C^{k,\ell}(u)$$

$$\psi_r^{k,\ell}(u,y_r) = \frac{r_C \, r_{\bar{C}}}{r_{\bar{C}} - iu} \, \phi_s(u) \, \psi_C^{k,\ell}(u) e^{iuy_r - \int_0^{y_r} \lambda_r(\xi)d\xi}$$

$$\psi_r^{k,\ell}(u) = \frac{r_C \, r_{\bar{C}}}{r_{\bar{C}} - iu} \, \phi_s(u) \left(\frac{\phi_r(u) - 1}{iu}\right) \psi_C^{k,\ell}(u)$$

all k, ℓ except $k = K$ and $\ell = L$

$$P(A)^{k,\ell} \, \psi_A^{k,\ell}(u) = p_A r_A \, \psi_C^{k-1,\ell} \ , \quad k \geq 1, \text{ all } \ell$$

$$P(B)^{k,\ell} \, \psi_B^{k,\ell}(u) = p_B r_B \, \psi_C^{k,\ell-1}(u), \quad \ell \geq 1, \text{ all } k$$

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$$P(AB) = q_A^K \, q_B^L \ .$$

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C. DISPLACEMENT (SUPPRESSION)

On both sides, on each round fired, either a total miss or a near-miss occurs. A near-miss either causes a displacement or a kill. During a displacement, a contestant is under fire and cannot return fire. Let

$$X_A \sim \text{ned}(r_A) \qquad \text{and} \qquad X_B \sim \text{ned}(r_B)$$

$T_{d_A}' = $ rv time for A to displace and resume firing.

Similarly for B.

$$T_{d_A} \sim \text{ned}(1/\delta_A) \qquad \text{and} \qquad T_{d_B} \sim \text{ned}(1/\delta_B)$$

$k_A = P[A \text{ kills} \mid \text{near miss by } A]$; Similarly for B

$o_A = P[A \text{ scores a near miss}]$; Similarly for B .

W & A2
$$P[A] = \frac{k_A o_A r_A (1 + o_A r_A \delta_B)(1 + k_B o_B r_B \delta_A)}{k_A o_A r_A (1 + o_A r_A \delta_B)(1 + k_B o_B r_B \delta_A) + k_B o_B r_B (1 + o_B r_B \delta_A)(1 + k_A o_A r_A \delta_B)} \; .$$

XII. TIME-OF-FLIGHT INCLUDED

$$\left.\begin{array}{l} \text{Let } T_{Fi} - \text{rv, } i\text{'s time-of-flight} \\[4pt] T_{Ki} - \text{rv, } i\text{'s time-to-fire killing round} \\[4pt] T_i - \text{rv, } i\text{'s time-to-hit} \\[4pt] \phi_{Fi}(u) - \text{cf of } i\text{'s pdf of } T_{Fi} \\[4pt] \phi_{Ki}(u) - \text{cf of } i\text{'s pdf of } T_{Ki} \\[4pt] \phi_i(u) - \text{cf of } i\text{'s pef of } T_i \end{array}\right\} \quad i = A, B$$

A. NO-DELAY DUEL

Each contestant fires as rapidly as possible (no waiting for round in air to land).

1. T_{FA}, T_{FB} rv's

$$\Phi_{Ki}(u) = \frac{p_i \phi_i(u)}{1 - q_i \phi_i(u)} \; ; \quad \Phi_i(u) = \Phi_{Ki}(u) \, \phi_{Fi}(u); \quad i = A, B$$

$$P(A) = \frac{1}{2} + \frac{1}{2\pi i} \; (P) \int_{-\infty}^{\infty} \Phi_A(-u) \, \Phi_{KB}(u) \, \frac{du}{u}$$

$$= \frac{1}{2\pi i} \int_{L} \Phi_A(-u) \, \Phi_{KB}(u) \, \frac{du}{u}$$

$$= 1 + \frac{1}{2\pi i} \int_{U} \Phi_A(-u) \, \Phi_{KB}(u) \, \frac{du}{u}$$

$$P(AB) = \frac{1}{2\pi i} \; (P) \int_{-\infty}^{\infty} [\Phi_{KA}(-u) \, \Phi_B(u) - \Phi_{KB}(u) \, \Phi_A(-u)] \, \frac{du}{u}$$

$$= \frac{1}{2\pi i} \int_{L \, or \, U} [\Phi_{KA}(-u) \, \Phi_B(u) - \Phi_{KB}(u) \, \Phi_A(-u)] \, \frac{du}{u}$$

Example: Let $X_A \sim \text{ned}(r_A)$ and $X_B \sim \text{ned}(r_B)$

$T_{FA} \sim \text{ned}(1/\tau_A)$ and $T_{FB} \sim \text{ned}(1/\tau_B)$

$$P(A) = \frac{1}{1 + \tau_A p_B r_B} \left(\frac{p_A \, r_A}{p_A r_A + p_B r_B} \right)$$

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$$P(AB) = \frac{p_A r_A \, p_B r_B [\tau_B (1 + \tau_A \, p_A r_A) + \tau_A (1 + \tau_B \, p_B r_B)]}{(1 + \tau_B \, p_A r_A)(1 + \tau_A \, p_B r_B)(p_A r_A + p_B r_B)} \; .$$

2. $\underline{T_{FA} = \tau_A, \quad T_{FB} = \tau_B \quad \text{TOF's} \quad \tau_A, \tau_B \quad \text{Constants}}$

$$P(A) = \frac{1}{2} + \frac{1}{2\pi i} \; (P) \int_{-\infty}^{\infty} e^{-i\tau_A u} \, \Phi_{KA}(-u) \, \Phi_{KB}(u) \; \frac{du}{u}$$

$$= \frac{1}{2\pi i} \int_{L} e^{-i\tau_A u} \, \Phi_{KA}(-u) \, \Phi_{KB}(u) \; \frac{du}{u}$$

$$P(AB) = \frac{1}{2\pi i} \; (P) \int_{-\infty}^{\infty} \Phi_{KA}(-u) \, \Phi_{KB}(u) \left[e^{i\tau_B u} - e^{-i\tau_A u} \right] \frac{du}{u}$$

$$= \frac{1}{2\pi i} \int_{U} e^{i\tau_B u} \, \Phi_{KA}(-u) \, \Phi_{KB}(u) \; \frac{du}{u}$$

$$- \frac{1}{2\pi i} \int_{L} e^{-i\tau_A u} \, \Phi_{KB}(-u) \, \Phi_{KA}(u) \; \frac{du}{u} \; .$$

<u>Example</u>: Let $X_A \sim ned(r_A)$ and $X_B \sim ned(r_B)$.

$$P(A) = \frac{p_A r_A \, \exp[-p_B r_B \, \tau_A]}{p_A r_A + p_B r_B}$$

$$P(AB) \;=\; \frac{p_A r_A (1 - \exp[- p_B r_B \tau_A]) + p_B r_B (1 - \exp[- p_A r_A \tau_B])}{p_A r_A + p_B r_B} \;.$$

B. DUEL WITH DELAY

Each contestant waits until his last round fired has landed before he prepares and fires his next round. The general solutions are the same as for the no delay case (Section A, above), except:

$$\phi_{KA}(u) \;=\; \frac{p_A \phi_A(u)}{1 - q_A \phi_A(u)\, \phi_{FA}(u)} \quad \text{and} \quad \phi_{KB}(u) \;=\; \frac{p_B \phi_B(u)}{1 - q_B \phi_B(u)\, \phi_{FB}(u)} \;.$$

Example 1: Let $X_A \sim \text{ned}(r_A)$ and $X_B \sim \text{ned}(r_B)$

$$T_{FA} \sim \text{ned}(1/\tau_A) \quad \text{and} \quad T_{FB} \sim \text{ned}(1/\tau_B) \;.$$

$$P(A) \;=\; \frac{\left[\begin{array}{c} p_A r_A [p_A r_A \tau_B^2 - p_B r_B \tau_A \tau_B \\[4pt] - (\tau_B + r_A \tau_A \tau_B + \tau_A + r_B \tau_A \tau_B)(p_B r_B \tau_B - 1 - r_B \tau_B] \end{array}\right]}{\left[\begin{array}{c} (p_A r_A \tau_B - p_B r_B \tau_A)^2 + (\tau_B + r_A \tau_A \tau_B + \tau_A + r_B \tau_A \tau_B) \\[4pt] \cdot [p_A r_A (1 + r_B \tau_B) + p_B r_B (1 + r_A \tau_A)] \end{array}\right]}$$

Example 2: Let $X_A \sim \text{ned}(r_A)$ and $X_B \sim \text{ned}(r_B)$

$$T_{FA} = \tau_A \text{ (fixed)} \quad \text{and} \quad T_{FA} = 0$$

$$P(A) \;=\; \frac{p_A r_A \exp[-p_B r_B \tau_A]}{r_A(1 - q_A \exp[-p_B r_B \tau_A]) + p_B r_B}$$

$$P(AB) = \frac{p_A r_A (1 - \exp[-p_B r_B \tau_A])}{r_A (1 - q_A \exp[-p_B r_B \tau_A]) + p_B r_B} \quad .$$

C. A MIXED PROCEDURE

Let $T_{FA} = \tau_A$, $T_{FB} = \tau_B$; X_A, X_B are rv's, where A uses the delay procedure and B uses the no-delay procedure.

$$\phi_{KA}(u) = \frac{p_A \phi_A(u)}{1 - q_A \phi_A(u) \exp[i\tau_A u]}$$

$$\phi_{KB}(u) = \frac{p_B \phi_B(u)}{1 - q_B \phi_B(u)}$$

$P(B)$, $P(AB)$ have essential singularities.

$$P(A) = \frac{1}{2\pi i} \int_L e^{-i\tau_A u} \phi_{KA}(-u) \phi_{KB}(u) \frac{du}{u} \quad .$$

Example: Let $X_A \sim ned(r_A)$ and $X_B \sim ned(r_B)$.

$$P(A) = \frac{p_A r_A e^{-p_B r_B \tau_A}}{r_A (1 - q_A e^{-p_B r_B \tau_A}) + p_B r_B} \quad .$$

D. SPECIAL CASE WHERE TIME-OF FLIGHT VARIES LINEARLY WITH TIME

This is a NO-DELAY PROCEDURE, where X_A, X_B are rv's and T_{FA}, T_{FB} are linearly varying (see Figure below for $T_{FA} = \tau_A$) deterministic

variables. A similar situation for T_{FB} obtains.

1. **Linearly Increasing Time-of-Flight**
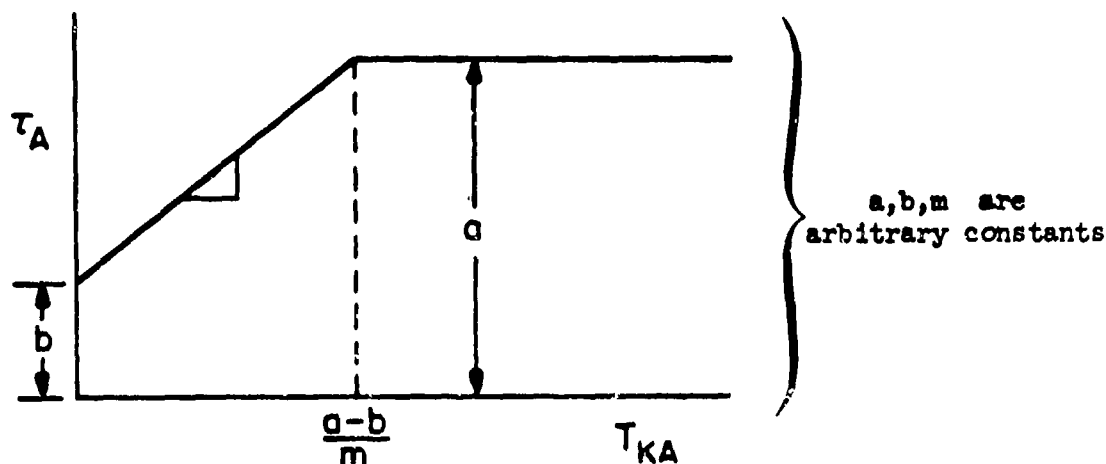
$$P(A) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-iau} \Phi_{KA}(-u) \left[\Phi_{KB}(u) - 1 \right] \frac{du}{u}$$

$$+ \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \frac{\{ \exp [-i (\frac{a - b}{m})u] - 1 \}}{u}$$

$$\cdot \left(\int_{-\infty}^{\infty} \frac{[\Phi_{KB}(w) - 1]}{w} \left\{ e^{-ibw} \Phi_{KA}[u - (1 + m)w] - e^{-iaw} \Phi_{KA}(u - w) \right\} dw \right) du$$

P(B) is obtained by interchanging B and A and replacing a,b,m by say, c,d,n.

Example: Let $X_A \sim \text{ned}(r_A)$ and $X_B \sim \text{ned}(r_B)$.

$$P(A) = p_A r_A \exp [- b\, p_B r_B] \left\{ \exp \left[-\left(\frac{a - b}{m} \right) \left[p_A r_A + (1 + m)p_B r_B \right] \right] \right.$$

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$$\cdot \left(\frac{1}{p_A r_A + p_B r_B} - \frac{1}{p_A r_A + (1+m)p_B r_B} \right) + \frac{1}{p_A r_A + (1+m)p_B r_B} \right\} .$$

Letting $a \to \infty$,

$$P(A) = \frac{p_A r_A \exp[-b\, p_B r_B]}{p_A r_A + (1 + m)p_B r_B} \quad .$$

2. **Linearly Decreasing Time-of-Flight**

From Section 1. above, $P(A)$ also applies to this case. Just replace m by $-m$. The constant a must be $+$ or zero.

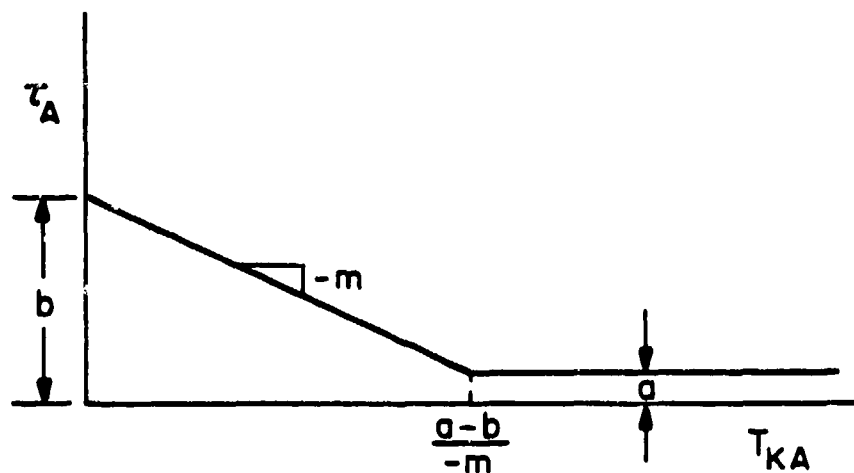**Example:** Let $X_A \sim \text{ned}(r_A)$, $X_B \sim \text{ned}(r_B)$, and $a = 0$.

$$P(A) = p_A r_A \exp[-b\, p_B r_B]\left\{ \exp\left[-\left(\frac{b}{m}\right)\left[p_A r_A + (1 - m)p_B r_B\right]\right]\right.$$

$$\left. \cdot \left(\frac{1}{p_A r_A + p_B r_B} - \frac{1}{p_A r_A + (1 - m)p_B r_B}\right) + \frac{1}{p_A r_A + (1 - m)p_B r_B}\right\} \quad .$$

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E. **LIMITED-AMMUNITION, RANDOM INDEPENDENT AMMUNITION RESUPPLY**

Delay Procedure (IFT and TOF Alternate)

Let k_0, ℓ_0 - A and B's initial ammunition supply (fixed)

$\qquad k, \ell$ - A and B's ammunition replenishment supply

$\qquad\qquad$ (fixed and same for each resupply).

Replenishments arrive randomly and independently of the firing process. Resupply inter-arrival times, T_{WA}, T_{WB}, are $\text{ned}(r_{WA}, r_{WB})$. The subscript F refers to T_F (TOF).

$$\Phi_{KA}(u) = p_A \left\{ \frac{1 + iuq_A^{k_0} \left[\dfrac{C_1^{k_0}(u)}{r_{WA}[1 - q_A^k C_1^k(u)] - iu} \right]}{1 - q_A \Phi_{FA}(u)\, \phi_A(u)} \right\}$$

$\phi_A(u) = \Phi_{KA}(u)\, \phi_{FA}(u)$, where

$$C_1(u) = \phi_{FA}\{r_{WA}[1 - q_A^k C_1^k(u) - iu\}\, \phi_A\{r_{WA}[1 - q_A^k C_1^k(j)] - iu\}$$

$$P(A) = \frac{1}{2\pi i} \int_L \Phi_A(-u)\, \Phi_{KB}(u)\, \frac{du}{u}$$

$$P(AB) = 1 - P(A) - P(B) \ .$$

Special Case: Zero flight time.

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$$\Phi_{KA}(u) = \Phi_A(u) = p_A \phi_A(u) \left\{ \frac{1 + \dfrac{iu(q_A^{k_O} C_2^{k_O}(u))}{r_{WA}[1 - q_A^k C_2^k(u)] - iu}}{1 - q_A \phi_A(u)} \right\}$$

where

$$C_2(u) = \phi_A\{r_{WA}[1 - q_A^k C_2^k(u)] - iu\} \quad .$$

Example: Let $X_A \sim \text{ned}(r_A)$, $X_B \sim \text{ned}(r_B)$, where B has unlimited ammunition, and let

$$[q_A C_1(u)]^k \simeq 0$$

$$\Phi_A(u) = \frac{p_A r_A}{p_A r_A - iu} \left\{ 1 + \frac{iu}{r_{WA} - iu} \left[\frac{q_A r_A}{r_A + r_{WA} - iu} \right]^{k_O} \right\}$$

$$\Phi_B(u) = \frac{p_B r_B}{p_B r_B - iu}$$

J & Bl $\quad F(A) = \dfrac{p_A r_A}{p_A r_A + p_B r_B} \left[1 - \dfrac{p_B r_B}{p_B r_B + r_{WA}} \left(\dfrac{r_A q_A}{r_A + r_{WA} + p_B r_B} \right)^{k_O} \right] \quad .$

XIII. BURST FIRING

A. TIME BETWEEN ROUNDS IN A BURST IS A RV

Let z_i - fixed number of rounds i fires in a burst

T_i - rv, time-between-bursts

T_{Gi} = rv, time-between-rounds in i's bursts

$\phi_i(u)$ = cf of T_i

$\phi_{Gi}(u)$ = cf of T_{Gi}

where i = A,B.

$$\Phi_A(u) = \frac{p_A \phi_A(u)\left[1 - (q_A \phi_{GA}(u))^{z_A}\right]}{\left[1 - q_A \phi_{GA}(u)\right]\left[1 - q_A^{z_A} \phi_A(u) \phi_{GA}^{z_A-1}(u)\right]}$$

$$P(A) = \frac{1}{2\pi i} \int_L \Phi_A(-u) \Phi_B(u) \frac{du}{u} .$$

Example 1: Let $T_A \sim$ ned(ρ_A) and $T_B \sim$ ned(ρ_B)

$T_{GA} \sim$ ned(r_A) and $T_{GB} \sim$ ned(r_B)

$q_A^{z_A} \simeq 0$ and $q_B^{z_B} \simeq 0$

$$P(A) = \frac{p_A \rho_A \left\{ \begin{array}{l} (p_A r_A + \rho_A + p_B r_B + \rho_B)[r_A r_B(p_B r_B + \rho_A) + (r_B - r_A)p_B r_B \rho_B] \\ + (r_A r_B + p_B r_B \rho_B) + (p_A r_A \rho_A - p_B r_B \rho_B) \end{array} \right\}}{r_B \left\{ \begin{array}{l} (p_A r_A \rho_A - p_B r_B \rho_B)^2 + (p_A r_A + \rho_A)(p_B r_B + \rho_B)(p_A r_A \rho_A + p_B r_B \rho_B) \\ + p_A r_A \rho_A(p_B r_B + \rho_B)^2 + p_B r_B \rho_B(p_A r_A + \rho_A)^2 \end{array} \right\}}$$

Example 2: B does not fire in bursts. Let $T_A \sim$ ned(ρ_A); $T_{GA} \sim$ ned(r_A) and $X_B \sim$ ned(r_B). Also let $q_A^{z_A} \simeq 0$.

Bh3(2) $P(A) = \dfrac{p_A \rho_A (r_A + p_B r_B)}{(\rho_A + p_B r_B)(p_A r_A + p_B r_B)}$.

B. TIME BETWEEN ROUNDS IN A BURST IS CONSTANT

A (burst firer)	B (no bursts)

$T_A = rv$, time-between bursts $X_B = rv$

T_{GA} = time-between-rounds in a
    burst = a (constant)

z = number of rounds in a burst
    (fixed)

$f_A(t)$ = pdf of T_A

$\phi_A(u)$ = cf of $f_A(t)$

$$P(A) = \frac{1}{2\pi i} \int_L \frac{p_A \phi_A(-u)[1 - q_A^z \exp(-iauz)]\, p_B \phi_B(u)\,du}{[1 - q_A \exp(-iau)][1 - q_A^z \phi_A(-u)\exp(-iau(z-1))](1 - q_B \phi_B(u))u} .$$

Example 1: Let $T_A \sim ned(\rho_A)$ and $X_B \sim ned(r_B)$.

$$P(A) = \frac{p_A \rho_A [1 - q_A^z \exp(-za\, p_B r_B)]}{[1 - q_A \exp(-a p_B r_B)][\rho_A + p_B r_B - q_A^z \rho_A \exp(-(z-1)a p_B r_B)]} .$$

Example 2: Same as Example 1, except let $z = 3$.

$$P(A) = \frac{\rho_A\, a\, p_A \{1 - [q_A \exp(-a p_B r_B)]^3\}}{[1 - q_A \exp(-a p_B r_B)]\{a p_B r_B + \rho_A a[1 - q_A(q_A \exp(-a p_B r_B))^2]\}} .$$

Consider three dimensionless parameters:

(1) a $p_B r_B$ (B's hit rate / A's firing rate between rounds in a burst)

(2) a o_A (A's rate between bursts / A's firing rate between rounds in a burst)

(3) p_A .

(See following plotted curves on next page).   A5

XIV. MULTIPLE WEAPONS

A.   VOLLEY FIRE (ALL WEAPONS FIRED SIMULTANEOUSLY)

1.   Unlimited Ammunition

Ammunition fired in volleys of v and w rounds by A and B,
respectively. Let

p_A = P[Volley by A hits]

X_A = rv, IFT between A's volleys

d_A = P[a given round in A's volley kills | A's volley hits]. This
is the same for all rounds in a volley and, all are independent.

With similar notation for B.

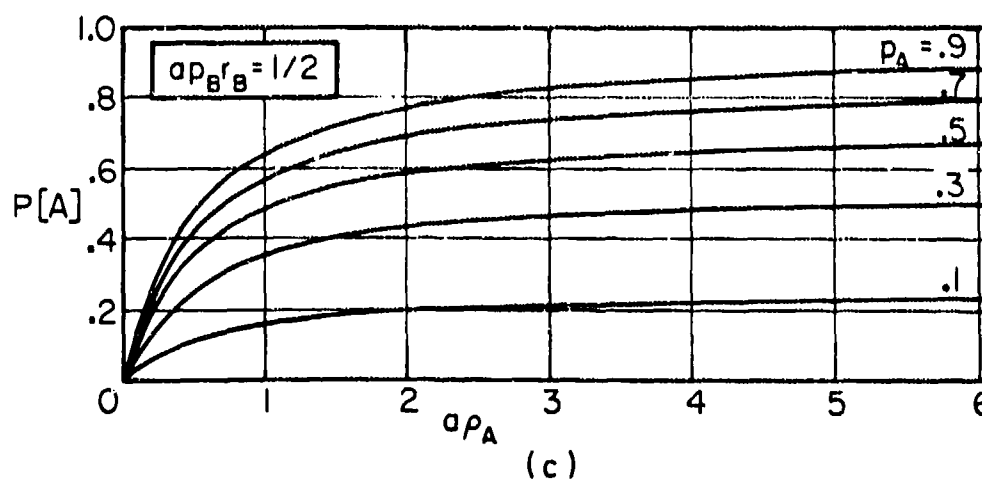$$P(A) = \frac{1}{2\pi i} \int_L \Phi_A(-u) \, \Phi_B(u) \, \frac{du}{u}$$

where

$$\Phi_A(u) = \frac{[1 - (1 - d_A)^v]p_A \phi_A(u)}{1 - [q_A + (1 - d_A)^v p_A]\phi_A(u)} .$$

Example: Let $X_A \sim ned(r_A)$ and $X_B \sim ned(r_B)$.   Kw & El

C217

(a)

(b)

(c)
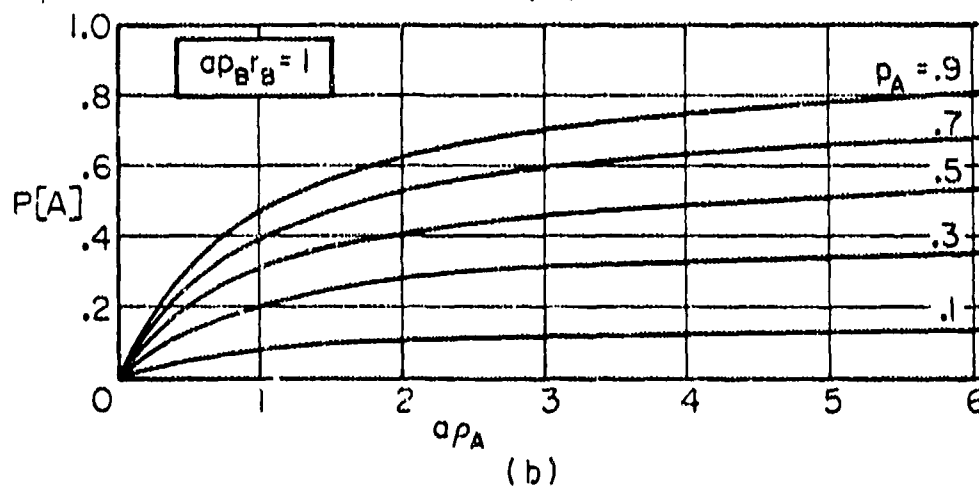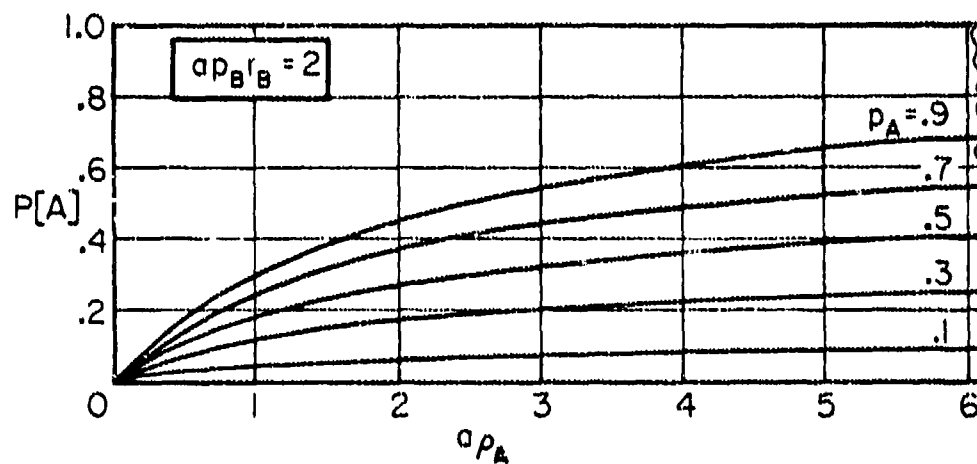
Burst Firer Versus Random-Firer
(Both ned)

A5

C218

$$P(A) = \frac{p_A r_A [1 - (1 - d_A)^V]}{p_A r_A [1 - (1 - d_A)^V] + p_B r_B [1 - (1 - d_B)^W]} \quad .$$

2. Limited Ammunition

Let $P[I = i] = \alpha_i$ and $P[J = j] = \beta_j$ for $i, j = 0, 1, 2, \ldots$

$$P(A) = \frac{1}{2\pi i} \int_L \Phi_{A1}(-u) \, \Phi_{B1}(u) \, \frac{du}{u} + P[\bar{\bar{H}}_B][1 - P[\bar{\bar{H}}_A]]$$

$$\Phi_{A1}(u) = \sum_{i=1}^{\infty} \left[\frac{1 - (1 - d_A)^V}{(1 - d_A)^V} \right] \left[\frac{(1 - d_A)^V \, p_A \phi_A(u)}{1 - q_A \phi_A(u)} \right]^i$$

$$\cdot \left\{ \sum_{j=1}^{\infty} \alpha_i (1 - I_{q_A \phi_A(u)}(j - i + 1, i) \right\}$$

$$P[\bar{\bar{H}}_A] = \sum_{i=0}^{j-1} \alpha_i \sum_{i=j}^{\infty} \alpha_i \sum_{k=0}^{i-1} \binom{i}{k} p_A^k q_A^{i-k} [(1 - d_A)^V]^k \quad ,$$

and similarly for B.

$$P(AB) = P[\bar{\bar{H}}_A] \, P[\bar{\bar{H}}_B] \quad .$$

Example: Let $\alpha_i = (1 - \alpha)\alpha^i$ and $\beta_j = (1 - \beta)\beta^j$ for $i, j = 0, 1, 2, \ldots$ with $X_A \sim \text{ned}(r_A)$ and $X_B \sim \text{ned}(r_B)$. If d_A and d_B are sufficiently small that $(1 - d_A)^V \cong 1 - d_A v$ and $(1 - d_B)^W \cong 1 - d_B w$, then

FD - CRIFT

$$P(A) = \left\{ \frac{\alpha \, p_A r_A d_A \, v}{r_A[1 - \alpha(1 - p_A d_A v)] + r_B[1 - \beta(1 - p_B d_B w)]} \right\}$$

$$\cdot \left[\frac{\beta \, p_B d_B \, w}{1 - \beta(1 - p_B d_B w)} \right] + \left[\frac{1 - \beta}{1 - \beta(1 - p_B d_B w)} \right] \left[\frac{\alpha \, p_A d_A \, v}{1 - \alpha(1 - p_A d_A v)} \right]$$

Kw &
Bl

$$P(AB) = \left[\frac{1 - \alpha}{1 - \alpha(1 - p_A d_A v)} \right] \left[\frac{1 - \beta}{1 - \beta(1 - p_B d_B w)} \right] .$$

B. <u>MULTIPLE WEAPONS - FIRED RANDOMLY</u>

A has k weapons with $X_{Ai} \sim \text{ned}(r_{Ai})$ and kill probabilities p_{Ai}, $i = 1, 2, \ldots, k$

B has ℓ weapons with $X_{Bj} \sim \text{ned}(r_{Bj})$ and kill probabilities p_{Bj}, $j = 1, 2, \ldots, \ell$.

$$P[A, \text{kill by } i\text{-th weapon}] = \frac{p_{Ai} \, r_{Ai}}{\sum\limits_{i=1}^{k} p_{Ai} r_{Ai} + \sum\limits_{j=1}^{\ell} p_{Bj} r_{Bj}} .$$

$$P(A) = \frac{\sum\limits_{i=1}^{k} p_{Ai} \, r_{Ai}}{\sum\limits_{i=1}^{k} p_{Ai} r_{Ai} + \sum\limits_{j=1}^{\ell} p_{Bj} r_{Bj}} .$$

If $N_{Ai} = rv$, number of rounds of the i-th kind fired by A in making a kill, then

$$E[N_{Ai}, A] = \frac{p_{Ai}r_{Ai}\left[r_{Ai} + \sum\limits_{\nu=1, \nu\neq i}^{k} p_{a\nu}r_{a\nu} + \sum\limits_{j=1}^{\ell} p_{Bj}r_{Bj} \right]}{\left[\sum\limits_{i=1}^{k} p_{Ai}r_{Ai} + \sum\limits_{j=1}^{\ell} p_{Bj}r_{Bj} \right]^2}$$

$$V[N_{Ai}, A] = \frac{2r_{Ai}^2 p_{Ai}q_{Ai}\left[r_{Ai} + \sum\limits_{\nu=1, \nu\neq i}^{k} p_{A\nu}r_{A\nu} + \sum\limits_{j=1}^{\ell} p_{Bj}r_{Bj} \right]}{\left[\sum\limits_{i=1}^{k} p_{Ai}r_{Ai} + \sum\limits_{j=1}^{\ell} p_{Bj}r_{Bj} \right]^3} \qquad \text{Bhl}$$

C. **MULTIPLE WEAPONS - FIRED ALTERNATELY**

Each contestant fires two weapons alternately.

For A: Weapon 1 - k_1 rounds fired each cycle, each with kill probability

p_{A1} and IFT X_{A1}

Weapon 2 - k_2 rounds fired each cycle, each with kill probability

p_{A2} and IFT X_{A2}

For B: Weapon 1 - ℓ_1 rounds fired each cycle, each with p_{B1} and IFT X_{B1}

Weapon 2 - ℓ_2 rounds fired each cycle, each with p_{B2} and IFT X_{B2}.

Each contestant starts with Weapon 1 unloaded.

$$\Phi_A(u) = \frac{\left\{ \begin{array}{l} p_{A1}\phi_{A1}(u)\{1 - [q_{A1}\phi_{A1}(u)]^{k_1}\}[1 - q_{A2}\phi_{A2}(u)] + p_{A2}\phi_{A2}(u) \\ \cdot [q_{A1}\phi_{A1}(u)]^{k_1}\{1-[q_{A2}\phi_{A2}(u)]^{k_2}[1 - q_{A1}\phi_{A1}(u)] \end{array} \right\}}{\left\{ \begin{array}{l} [1 - q_{A1}\cdot\phi_{A1}(u)][1 - q_{A2}\cdot\phi_{A2}(u)] \\ \cdot \{1 - [q_{A1}\phi_{A1}(u)]^{k_1}[q_{A2}\phi_{A2}(u)]^{k_2}\} \end{array} \right\}}$$

for $\Phi_B(u)$, replace A by B, k_1 by ℓ_1, and k_2 by ℓ_2 .

$$P(A) = \frac{1}{2\pi i} \int_L \Phi_A(-u) \, \Phi_B(u) \, \frac{du}{u} \; .$$

Example 1: Let $X_{A1} \sim \text{ned}(r_{A1})$ and $X_{A2} \sim \text{ned}(r_{A2})$

$$X_{B1} \sim \text{ned}(r_{B1}) \quad \text{and} \quad X_{B2} \sim \text{ned}(r_{B2})$$

and $k_1 = k_2 = \ell_1 = \ell_2 = 1$.

$$\Phi_A(u) = \frac{r_{A1}r_{A2}(1 - q_{A1}q_{A2}) - i \, p_{A1}r_{A1} \, u}{-u^2 - iu(r_{A1} + r_{A2}) + r_{A1}r_{A2}(1 - q_{A1}q_{A2})} \; ,$$

and, similarly for $\Phi_B(u)$.

$$P(A) = \frac{\left\{ \begin{array}{l} r_{A1}r_{A2}(1 - q_{A1}q_{A2})[(r_{B1} + r_{B2})^2 + (r_{A1} + r_{A2})(r_{B1} + r_{B2}) \\[4pt] \quad + r_{A1}r_{A2}(1 - q_{A1}q_{A2}) - r_{B1}r_{B2}(1 - q_{B1}q_{B2})] \\[4pt] \quad + (r_{A1} + r_{A2} + r_{B1} + r_{B2})[r_{A1}p_{A1} \, r_{B1}r_{B2}(1 - q_{B1}q_{B2}) \\[4pt] \quad - r_{B1}p_{B1} \, r_{A1}r_{A2}(1 - q_{A1}q_{A2})] \\[4pt] \quad + r_{A1}r_{B1} \, p_{A1}p_{B1}[r_{B1}r_{B2}(1 - q_{B1}q_{B2}) \cdot r_{A1}r_{A2}(1 - q_{A1}q_{A2})] \end{array} \right\}}{\left\{ \begin{array}{l} [r_{A1}r_{A2}(1 - q_{A1}q_{A2}) - r_{B1}r_{B2}(1 - q_{B1}q_{B2})]^2 \\[4pt] \quad + (r_{A1} + r_{A2})(r_{B1} + r_{B2})[r_{B1}r_{B2}(1 - q_{B1}q_{B2}) + r_{A1}r_{A2}(1 - q_{A1}q_{A2})] \\[4pt] \quad + r_{A1}r_{A2}(1 - q_{A1}q_{A2})[(r_{B1} + r_{B2})^2 + r_{B1}r_{B2}(1 - q_{B1}q_{B2})(r_{A1} + r_{A2})^2] \end{array} \right\}}$$

Example 2: Let $\ell_1 = \ell_2 = 1$, A has only one weapon. Also let,

$$X_A \sim \text{ned}(r_{A1}), \quad X_{B1} \sim \text{ned}(r_{B1}) \quad \text{and} \quad X_{B2} \sim \text{ned}(r_{B2}) \; .$$

$$P(A) \; = \; \frac{p_A r_A (p_A r_A + r_{B1} + r_{B2} - r_{B1} p_{B1})}{(p_A r_A)^2 + p_A r_A (r_{B1} + r_{B2}) + r_{B1} r_{B2}(1 - q_{B1} q_{B2})} \; .$$

Example 3: Let $k_1 = k_2 = 1$, B has only one weapon. Also let,

$$X_{A1} \sim \text{ned}(r_{A1}), \; X_{A2} \sim \text{ned}(r_{A2}) \quad \text{and} \quad X_B \sim \text{ned}(r_B) \; .$$

$$P(A) \; = \; \frac{r_{A1} r_{A2}(1 - q_{A1} q_{A2}) + p_{A1} r_{A1} \, p_B r_B}{(p_B r_B)^2 + p_B r_B (r_{A1} + r_{A2}) + r_{A1} r_{A2}(1 - q_{A1} q_{A2})} \; . \qquad \text{Eh?}$$

D. MULTIPLE WEAPONS, EACH FIRED CONSECUTIVELY UNTIL FAILURE

1. Ammunition - Limitation

Let   A - k rounds initially   Let   B - ℓ rounds initially

     - m_A weapons   - m_B weapons

     $X_A \sim \text{ned}(r_A)$ $X_B \sim \text{ned}(r_B)$

     T_{LA} - time-to-failure, same T_{LB} - $\text{ned}(r_{LB})$
            for each weapon when
            in use $\sim \text{ned}(r_{LA})$

$$P(A) \; = \; \frac{1}{2\pi i} \int_L \; \Phi_{A1}(-u) \, \Phi_{B1}(u) \, \frac{du}{u} \; + \; \Phi_{B0}(0)[1 - \Phi_{A0}(0)]$$

$$P(AB) \; = \; \Phi_{A0}(0) \, \Phi_{B0}(0),$$

where

$$\Phi_{A0}(0) = q_A^k \, I_{\left(\dfrac{r_A}{r_A + r_{LA}}\right)}(k, m_A) + \left(\dfrac{r_{LA}}{p_A r_A + r_{LA}}\right)^{m_A} I_{\left(\dfrac{p_A r_A + r_{LA}}{r_A + r_{LA}}\right)}(m_A, k)$$

$$\Phi_{A1}(u) = \dfrac{p_A \, r_A}{p_A r_A - iu}$$

$$\cdot \left[1 - \left(\dfrac{q_A \, r_A}{r_A - iu}\right)^k I_{\left(\dfrac{r_A - iu}{r_A + r_{LA} - iu}\right)}(k, m_A) \right.$$

$$\left. - \left(\dfrac{r_{LA}}{p_A r_A + r_{LA} - iu}\right)^{m_A} I_{\left(\dfrac{p_A r_A + r_{LA} - iu}{r_A + r_{LA} - iu}\right)}(m_A, k) \right]$$

2. Unlimited Ammunition - Random Initial Supply of Weapons (M_A, M_B)

$$P[M_A = i] = \alpha_i \; ; \quad \sum_{i=0}^{\infty} \alpha_i = 1 ; \quad P[M_B = j] = \beta_j \, , \quad \sum_{j=0}^{\infty} \beta_j = 1 ; \quad \text{with}$$

$$X_A \sim \text{ned}(r_A) \qquad \text{and} \qquad X_B \sim \text{ned}(r_B)$$

$$T_{LA} \sim \text{ned}(r_{LA}) \qquad \text{and} \qquad T_{LB} \sim \text{ned}(r_{LB}) \; .$$

$$P(A) = \dfrac{1}{2\pi i} \int_L \Phi_{A1}(-u) \, \Phi_{B1}(u) \dfrac{du}{u} + \Phi_{B0}(0)[1 - \Phi_{A0}(0)]$$

$$P(AB) = \Phi_{A0}(0) \, \Phi_{B0}(0)$$

where

$$\phi_{A0}(0) = \sum_{i=0}^{\infty} \alpha_i \left(\frac{r_{LA}}{p_A r_A + r_{LA}} \right)^i$$

$$\phi_{A1}(u) = \frac{p_A r_A}{p_A r_A - iu} \left[1 - \sum_{i=0}^{\infty} \alpha_i \left(\frac{r_{LA}}{p_A r_A + r_{LA} - iu} \right)^i \right] .$$

<u>Example 1</u>: Let $\alpha_i = (1 - \alpha)\alpha^i$ and $\beta_j = (1 - \beta)\beta^j$.

$$P(A) = \frac{p_A r_A \alpha \{[r_{LA}(1 - \alpha) + p_A r_A] + (1 - \beta)(p_B r_B + r_{LA})\}}{\{[r_{LA}(1 - \alpha) - p_A r_A][r_{LA}(1 - \alpha) + p_A r_A + r_{LB}(1 - \beta) + p_B r_B]\}}$$

$$P(AB) = \frac{(1 - \alpha)(1 - \beta)(p_A r_A + r_{LA})(p_B r_B + r_{LB})}{[r_{LA}(1 - \alpha) + p_A r_A][r_{LB}(1 - \beta) + p_B r_B]} .$$

<u>Example 2</u>:

For A: m_1 weapons (fixed)   For B: 1 weapon (no failures)

$$X_A \sim \text{ned}(r_A) \qquad\qquad X_A \sim \text{ned}(r_B)$$

$$T_{LA} \sim \text{ned}(r_{LA})$$

$$P(A) = \frac{p_A r_A}{p_A r_A + p_B r_B} \left[1 - \left(\frac{r_{LA}}{r_{LA} + p_A r_A + p_B r_B} \right)^{m_A} \right]$$

$$P(AB) = 0 .$$

Bh6

XV. MARKOV-DEPENDENT FIRE

See FD - FIFT for notation (pages C97 and 98).

A. POSITIVELY CORRELATED FIRE

A fires with positive correlation between hits

B fires with independent hit and kill probabilities

A is a three-state firer, $(\overline{H}, H\overline{K}, K)$

If $P[H_i \mid H_{i-1}] \triangleq p_0$, $p[H_i \mid \overline{H}_{i-1}] \triangleq p_1$, $p_0 > p_1$,

and

$$P[K_i \mid H_i] \triangleq p_k, \quad \text{for all } i.$$

Then

$$P[H_i] = \frac{p_1}{1 - p_0 + p_1} \qquad ; \qquad o = \text{Corr } [H_i, H_{i-1}] = p_0 - p_1 \ .$$

$$\phi_A(u) = \frac{p_1 p_k}{1 - p_0 + p_1}$$

$$\cdot \left(\frac{\phi_A(u)[1 - (p_0 - p_1) \, \phi_A(u)}{1 - \phi_A(u)[1 - p_1 + p_0(1 - p_k)(1 - \phi_A(u))] + p_1(1 - p_k) \, \phi_A(u)} \right);$$

B is the firer with independent hit and kill probabilities, i.e.,

$$P[K_i \mid H_i] = 1, \quad P[H_i] = p_B \text{ for all } i \ .$$

$$\phi_B(u) = \frac{p_B \phi_B(u)}{1 - q_B \phi_B(u)}$$

$$P(A) = \frac{1}{2\pi i} \int_L \Phi_A(-u) \, \Phi_B(u) \, \frac{du}{u} \quad .$$

<u>Example</u>: Let $X_A \sim \text{ned}(1)$ and $X_B \sim \text{ned}(r_B)$.

$$\Phi_A(u) = -\frac{C_1 C_2}{C_3} \frac{iu - C_3}{(iu - C_1)(iu - C_2)}$$

where

$$C_1 = \frac{1}{2}\left[1 + p_1 - p_0(1 - p_k) + \sqrt{1 + p_1 - p_0(1 - p_k)^2 - 4p_1 p_k} \right]$$

$$C_2 = \frac{1}{2}\left[1 + p_1 - p_0(1 - p_k) - \sqrt{1 + p_1 - p_0(1 - p_k)^2 - 4p_1 p_k} \right]$$

$$C_3 = 1 - p_0 + p_1 \quad .$$

$$\Phi_B(u) = \frac{C_4}{C_4 - iu}$$

where $\qquad C_4 = p_B r_B$

$$P(A) = \frac{C_1 C_2 (C_3 + C_4)}{C_3 (C_1 + C_4)(C_2 + C_4)} \quad .$$

Fil

B. <u>IFT'S ARE STATE-DEPENDENT AND ned</u>

For A: pdf IFT when in state $E_i = r_{Ai} \, e^{-r_{Ai} t}$, $t \geq 0$

$\qquad\qquad\qquad\qquad = \quad 0 \qquad$, elsewhere

for $i = 1, 2, \ldots, m$.

$$\underset{\sim}{D}_A(r_{Ai}) = \begin{pmatrix} r_{A0} & & & & & \\ & r_{A1} & & & 0 & \\ & & \ddots & & & \\ & & & r_{Ai} & & \\ & 0 & & & \ddots & \\ & & & & & r_{Am} \end{pmatrix}$$

$$\underset{\sim}{A} = \underset{\sim}{D}_A(r_{Ai})(\underset{\sim}{S}_A - \underset{\sim}{I}), \qquad i = 1,2,\ldots,m \ ,$$

$\lambda_i =$ i-th characteristic value of $\underset{\sim}{A}$, $\lambda_0 = 0$, $\lambda_i < 0$ ($i \neq 0$) for

$\quad i = 1,2,\ldots,m$.

If $\underset{\sim}{A}$ has $m+1$ linearly independent characteristic vectors, then

let $\underset{\sim}{X}$ be the matrix of characteristic vectors of $\underset{\sim}{A}$

$\quad \underset{\sim}{x}_0^T$ be the zero-th row of $\underset{\sim}{X}$

$\quad \underset{\sim}{x}_m'$ be the m-th column of $\underset{\sim}{X}^{-1}$

and $\underset{\sim}{x}_{0,m}$ be the vector whose i-th component is the product of the

$\quad$ i-th component of $\underset{\sim}{x}_0^T$ with the i-th component of $\underset{\sim}{x}_m'$.

For B: Interchange A and B, m and ℓ

$\quad$ X and Y (for B), x and y (for B), E and F (for B),

$\quad$ and λ and Θ (for B).

Now let $\underset{\sim}{Z} =$ an $m \times \ell$ matrix, whose ij-th component is $(\lambda_i/(\lambda_i + \Theta_j))$

1. **FD**

$$P(A) = 1 - \int_0^\infty \underset{\sim}{m}_A^T \, e^{\underset{\sim}{A}t} \, \underset{\sim}{A} \, \underset{\sim}{n}_A \, \underset{\sim}{m}_B^T \, e^{\underset{\sim}{B}t} \, \underset{\sim}{n}_B \, dt \ =$$

$$= \int_0^\infty \int_t^\infty \underset{\sim}{m}_A^T \underset{\sim}{e}^{At} A \underset{\sim}{n}_A \underset{\sim}{m}_B^T \underset{\sim}{e}^{Bt} \underset{\sim}{B} \underset{\sim}{n}_B \, d\tau \, dt$$

$$P(A) = \underset{\sim\sim}{x}_{\ell m}^T Z \underset{\sim\sim}{y}_{\ell\ell}$$

$$P(B) = 1 - P(A) \; .$$

2. FD with Fixed Surprise-Time

Let A have τ units of time before B acquires A and commences
the FD.

$$P(A) = \int_0^\tau \underset{\sim}{m}_A^T \underset{\sim}{e}^{At} A \underset{\sim}{n}_A + \int_\tau^\infty \int_t^\infty \underset{\sim}{m}_A^T \underset{\sim}{e}^{At} A \underset{\sim}{n}_A \underset{\sim}{m}_B^T \underset{\sim}{e}^{Bt(x-\tau)} \underset{\sim}{B} \underset{\sim}{n}_B \, dx \, dt$$

$$= \underset{\sim}{x}_Q^T (\underset{\sim}{D}(e^{\lambda_1 \tau}) - I) \underset{\sim}{x}_m' + \underset{\sim\sim}{x}_{\ell m}^T W \underset{\sim\sim}{y}_{\ell\ell}$$

where $\underset{\sim}{W}$ = an $m \times \ell$ matrix, whose ij-th component is $(\lambda_1 e^{\lambda_1 t}/\lambda_1 + \Theta_j)$

$$P(B) = 1 - P(A).$$

3. FD with Random Initial Surprise

$$\text{Let} \quad f_{T_S}(t) = \frac{1}{C} e^{-t/C} , \qquad t \geq 0, \; C > 0$$

$$= \quad 0 \quad , \qquad t < 0$$

$$P(A) = \underset{\sim}{x}_Q^T \underset{\sim}{D} \left(\frac{C\lambda_1}{1 - C\lambda_1} \right) \underset{\sim}{x}_m' + \frac{1}{C} \underset{\sim\sim}{x}_{\ell m}^T U \underset{\sim\sim}{y}_{\ell\ell} , \qquad \text{where}$$

Ba3 U = an $m \times \ell$ matrix, whose ij-th component is $\left(\dfrac{c\lambda_i}{(\lambda_i + \vartheta_j)(1 - c\lambda_i)} \right)$.

C. **CONTESTANT FIRING ORDER AND IFT'S ARE MARKOV-DEPENDENT**

FD - CRIFT

Let p_{ij} = P[j fires next, given i fired last], i,j = A,B. Then the transition matrix is:

$$
\begin{array}{cc}
 & \begin{array}{cc} A & \quad B \end{array} \\
\begin{array}{c} A \\[2em] B \end{array} &
\left(\begin{array}{cc} p_{AA} & p_{AB} \\[2em] p_{BA} & p_{BB} \end{array} \right) .
\end{array}
$$

The IFT = X_A if A fired last ,

= X_B if B fired last .

Initially, at t = 0, start <u>as though</u> A had fired last

$$
\Psi(u,y_A) = \left\{ U(y_A) + \frac{q_A\,\phi_A(u)}{W} \left[p_{AA} - q_B \phi_B(u)(p_{AA}p_{BB} - p_{AB}p_{BA}) \right] \right\}
$$

$$
\cdot\ e^{\,iuy_A - \int_0^{y_A} \lambda_A(t)dt}
$$

$$
\Psi(u,y_B) = \frac{p_{AB}\,p_B \phi_A(u)}{W}\ e^{\,iuy_B - \int_0^{y_B} \lambda_B(t)dt}
$$

$$\Psi_A(u) = \frac{p_A}{W}\left[p_{AA}\phi_A(u) - p_B(p_{AA}p_{BB} - p_{AB}p_{BA})\phi_A(u)\phi_B(u)\right],$$

$$\Psi_B(u) = (p_A/W)p_{AB}\phi_A(u), \quad \text{where}$$

$$W = 1 - p_{AA}q_A\phi_A(u) - p_{BB}q_B\phi_B(u) + q_Aq_B(p_{AA}p_{BB} - p_{AB}p_{BA})\phi_A(u)\phi_B(u)$$

$$P(A) = \frac{p_A[p_{AA} - q_B(p_{AA}p_{BB} - p_{AB}p_{BA})]}{1 - q_Ap_{AA} - q_Bp_{BB} + q_Aq_B(p_{AA}p_{BB} - p_{AB}p_{BA})}$$

$$P(B) = \frac{p_B\,p_{AB}}{1 - q_Ap_{AA} - q_Bp_{BB} + q_Aq_B(p_{AA}p_{BB} - p_{AB}p_{BA})} \quad . \qquad Bh3(1)$$

D. MULTIPLE WEAPONS

Each Contestant has Two Weapons – Fired in a Random Markov-Dependent Order

	A		B	
Weapon 1:	IFT = X_{A1},	p_{A1}	IFT = X_{B1},	p_{B1}
2:	= X_{A2},	p_{A2}	= X_{B2},	p_{B2}

Weapon Firing Order Transition Matrices

$$\alpha_{i,j} = P[A \text{ fires weapon } i \text{ next} \mid A \text{ fired weapon } j \text{ last}], \quad i,j = 1,2$$

$$\underset{\sim}{F_A} = \frac{1}{2}\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \qquad\qquad \underset{\sim}{F_B} = \frac{1}{2}\begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}\ .$$

In each case, firing starts with weapon 1. If

$$C = 1 - q_{A1}\alpha_{11}\,\phi_{A1}(u) - q_{A2}\alpha_{22}\phi_{A2}(u) - q_{A1}q_{A2}(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})\,\phi_{A1}(u)\,\phi_{A2}(u)$$

$$\phi_A(u) = \frac{\phi_{A1}(u)}{C}\left[p_{A1} + (p_{A1}q_{A2}\alpha_{22} - q_{A1}p_{A2}\alpha_{12})\,\phi_{A2}(u) \right] ,$$

and similarly for B.

$$P(A) = \frac{1}{2\pi i}\int_L \phi_A(-u)\,\phi_B(u)\,\frac{du}{u} .$$

Example: Let $X_{A1} \sim ned(r_{A1})$ and $X_{B1} \sim ned(r_{B1})$

$$X_{A2} \sim ned(r_{A2}) \quad\text{and}\quad X_{B2} \sim ned(r_{B2}) .$$

$$\phi_A(u) = \frac{c_1 c_2 - i\, p_{A1}r_{A1}\,u}{(c_1 - iu)(c_2 - iu)} , \qquad \text{where}$$

$$c_1 + c_2 = r_{A1}(1 - q_{A1}\alpha_{11}) + r_{A2}(1 - q_{A2}\alpha_{22})$$

$$c_1 c_2 = r_{A1}r_{A2}[1 - q_{A1}\alpha_{11} - q_{A2}\alpha_{22} + q_{A1}\alpha_{12}(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}) .$$

B is obtained similarly by replacing C, $c_1 c_2$ with D, $d_1 d_2$.

Bh8
$$P(A) = \frac{\left\{ \begin{array}{c} (c_1 + c_2 + d_1 + d_2)[p_{A1}r_{A1}\,d_1 d_2 - p_{B1}r_{B1}\,c_1 c_2 + c_1 c_2 (d_1 + d_2)] \\ + (c_1 c_2 - d_1 d_2)(c_1 c_2 - p_{A1}r_{A1}\,p_{B1}r_{B1}) \end{array} \right\}}{\left\{ \begin{array}{c} (c_1 c_2 - d_1 d_2)^2 + (c_1 + c_2)(d_1 + d_2)(c_1 c_2 + d_1 d_2) \\ + c_1 c_2 (d_1 + d_2)^2 + d_1 d_2 (c_1 + c_2)^2 \end{array} \right\}}$$

FUNDAMENTAL DUEL - MIXED INTERFIRING TIMES

(FD - MIFT)

Let $X_A = a_1$ (fixed) and $X_B = a$ rv, then

$$P(A) = \frac{1}{2} + \frac{1}{2\pi i} \ (P) \int_{-\infty}^{\infty} \phi_A(-u) \ \phi_B(u) \ \frac{du}{u}$$

$$= \frac{1}{2\pi i} \int_L \phi_A(-u) \ \phi_B(u) \ \frac{du}{u}$$

$$= \frac{1}{2\pi i} \int_L \frac{p_A \exp(-ia_1 u) \ p_B \phi_B(u) du}{[1 - q_A \exp(-ia_1 u)][1 - q_B \phi_B(u)]u} \quad .$$

Example: Let $X_B \sim ned(r_B)$.

$$P(A) = \frac{p_A \exp(-a_1 p_B r_B)}{1 - q_A \exp(-a_1 p_B r_B)}$$

$$= \frac{p_A \exp\left(-\frac{p_B r_B}{r_A}\right)}{1 - q_A \exp\left(-\frac{p_B r_B}{r_A}\right)}$$

where

$$\frac{1}{a_1} = r_A \quad .$$

Plots of this expression are given below.

(b)

(a)

(c)

A5 Fixed Fire Versus Exponential Random Fire